9

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.

Cambridge: DEIGHTON, BELL AND CO.
Leipzig: F. A. BROCKHAUS.
New York: MACMILLAN AND CO.

A TREATISE

ON THE

MATHEMATICAL THEORY

OF

ELASTICITY

BY

A. E. H. LOVE, M.A.

FELLOW AND LECTURER OF ST JOHN'S COLLEGE, CAMBRIDGE

VOLUME II.

CAMBRIDGE:
AT THE UNIVERSITY PRESS
1893 M.

Cambridge:

PRINTED BY C. J. CLAY, M.A. AND SONS,

AT THE UNIVERSITY PRESS.

PREFATORY NOTE.

I WISH to express my thanks to two friends who have given me much assistance in the work of proof-correction—Mr G. H. Bryan and Mr J. Larmor. The former has made a careful revision of the proofs, and the latter has given me several most valuable suggestions for the improvement of the more theoretical parts of the work. Prof. Greenhill has also kindly sent me a few corrections.

CONTENTS.

Historical Introduction.

CHAPTER XIII. THE BENDING OF RODS IN ONE PLANE.

CHAPTER XIV. THE BENDING AND TWISTING OF RODS OR WIRES IN THREE DIMENSIONS.

CHAPTER XV. GENERAL THEORY OF THIN RODS OR WIRES.

CHAPTER XVI. THEORY OF THE SMALL VIBRATIONS OF THIN RODS.

CHAPTER XVII. RESILIENCE AND IMPACT.

CHAPTER XVIII. GENERAL THEORY OF WIRES NATURALLY CURVED.

CHAPTER XIX. ELEMENTARY THEORY OF THIN PLANE PLATES.

CHAPTER XX. GENERAL THEORY OF THIN PLATES.

CHAPTER XXI. GENERAL THEORY OF THIN ELASTIC SHELLS.

CHAPTER XXII. APPLICATIONS OF THE THEORY OF THIN SHELLS.

CORRIGENDA.

p. 77, lines 5—7 from top, omit the words 'and there will be...twist'.

,, line 9 ,, , *for* $(B-C)$ *read* C.

Additional Corrigenda in Vol. I.

p. 74, scheme of transformation, *insert* z' in lowest left-hand compartment.

p. 131, line 10 from bottom, *for* ϕ_0 and ϕ_0 *read* ϕ_0 and $\dot{\phi}_0$.

,, line 11 ,, ,, , *for* $\frac{\sinh(at\nabla)}{at\nabla}\phi_0$ *read* $\frac{\sinh(at\nabla)}{at\nabla}\dot{\phi}_0$.

p. 185, line 9 from top, *for* a *read* a_1.

p. 343, line 2 from bottom, *for* $\frac{4B_2}{c^2\mu}$ *read* $\frac{2B_2}{c^2\mu}$.

p. 346, line 16 ,, ,, *for* luminated *read* laminated.

HISTORICAL INTRODUCTION.

In the first volume of this work we have seen how the Theory of Elasticity took its rise in an enquiry of Galilei's concerning the resistance of a beam to rupture by flexure; how the methods pursued in the 18th century in endeavours to answer this question led by a natural sequence to the discovery of the general equations of Elasticity; how this discovery was the prelude to a series of brilliant analytical researches concerned with problems of the most general character; and how, at the same time, it made possible an attempt to answer from mechanical principles questions of the deepest physical significance and the highest practical importance. The discussion of the number and meaning of the elastic constants threw light on the most recondite problems of "intermolecular force"; the laws of wave-propagation in solids, both by agreement and disagreement with optical experiments, illustrated the nature of the luminiferous medium; the theories of the torsion and flexure of beams supplied the engineer with valuable working formulæ. We saw that for practical application of these formulæ it was necessary always that the length of the beam should be great compared with its thickness—or that the linear dimensions of the body concerned should be of different orders of magnitude. We have now to consider in greater detail the special problems that arise when some linear dimensions of a body are small in comparison with others. These problems include the bending of rods, on which depends a large part of the theory of structures, the vibrations of bars, leading to theories of resilience and impact, the deformation of thin plates and its developments in regard to the vibrations of bells. Further, as we shall see hereafter, the series of difficult questions which turn upon elastic stability belong to the same class of problems.

3b

Theories of the behaviour of thin bodies, as of bodies in general, are of two kinds. Either they are founded on special hypotheses, or they start from the general equations of Elasticity. Before the time of Navier all writers on rods and plates naturally adopted the former method, after his time most valuable investigations will be found to proceed by way of the latter. His researches form the turning point in the history of special problems as in that of general theory. Nevertheless the persistence of the older theories after the discovery of the general equations had made more exact investigations possible, and even after they had been carried out, is one of the most noteworthy facts in the history of our subject[1].

We shall now trace briefly the development of the theory of thin rods and its applications, we shall then consider the theory of thin plates, and we shall conclude by noticing the theory of elastic stability.

We have already had occasion to state that the earliest problem attempted was that of the flexure of a beam. The stress across a section of the beam when bent was attributed to the extension and contraction of the fibres, and a series of investigations by Marriotte, James Bernoulli the elder, and others, culminating in the researches of Coulomb, were devoted to the foundation of a theory upon this assumption. Side by side with this theory of flexure, which attempted to estimate the stress from the character of the strain, we have a theory of rods regarded as strings which resist bending. This theory starts with the assumption of a couple across each normal section of the bent rod proportional to the curvature, and although the elder Bernoulli[2] deduced this from the notion of extension and contraction of the fibres, yet his investigation of the curve of the elastic central-line is actually the foundation of the later work of Euler and others.

A theory of the vibrations of thin rods is a natural outcome of researches on the flexure of beams. As soon as the notion of a flexural couple proportional to the curvature was established it could be noted that the work done in bending a rod is proportional to the square of the curvature. This fact was remarked by Daniel

[1] For example in M. Lévy's *Statique Graphique*, Paris, 1886, all the erroneous assumptions of the Bernoulli-Eulerian theory are reproduced, and made the basis of the theory of flexure.

[2] See I. p. 3.

Bernoulli[1], and the observation formed the foundation of Euler's theory[2] of the vibrations of rods—a theory which included the differential equation of lateral vibrations, and what we should now call the normal functions and the period-equation, with the six cases of terminal conditions when the ends are free, built-in, or simply supported. The method was what we should call variation of the energy-function. This step was suggested to Euler by Daniel Bernoulli[3] with a request that he would bring his knowledge of isoperimetric problems to bear upon it. In two memoirs published by Daniel Bernoulli[4] the differential equation and some of Euler's other results were given. The credit of these discoveries is therefore to be shared by these two mathematicians.

The suggestion of D. Bernoulli that the differential equation of the elastic line might be found by making a minimum the integral of the square of the curvature from end to end of the rod bore fruit in another direction. Starting from this suggestion Euler[5] was enabled to find the equation of the elastic line in the form in which it had been previously given by James Bernoulli the elder, and he proceeded to classify the forms of the curve. The curves obtained are those in which a thin rod can be held by forces and couples applied at the ends alone.

Euler[6] and afterwards Lagrange[7] worked at the problem of determining the least length of an upright column in order that it may not be bent by its own or an applied weight. These researches are the earliest in the region of elastic stability, and we shall consider them more fully later.

[1] In his xxvith letter to Euler (Oct. 1742). See Fuss, *Correspondance Mathématique et Physique*, St Pétersbourg, 1843, Tom. II.

[2] Given in the *Additamentum* 'De curvis elasticis' of the *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, 1744.

[3] In the letter above quoted.

[4] 'De vibrationibus...laminarum elasticarum......', and 'De sonis multifariis quos laminæ elasticæ...edunt...' published in *Commentarii Academiæ Scientiarum Imperialis Petropolitanæ*, XIII. 1751. The reader must be cautioned that with the writers of the 18th century a 'lamina' generally means a straight rod or curved bar, supposed to be cut out from a thin plate or shell by two normal sections near together. This usage lingers in many books, for example in Poisson's *Mécanique* and in Dr Besant's *Hydromechanics*.

[5] In the *Additamentum* 'De curvis elasticis' of the *Methodus inveniendi....*

[6] 'Sur la force des colonnes', *Hist. Acad. Berlin*, XIII. 1757.

[7] 'Sur la figure des colonnes', *Miscellanea Taurinensia*, V. 1773.

The flexure of a rod initially curved appears to have been first discussed by Euler[1], starting from the assumption that the flexural couple is proportional to the change of curvature. In this he was followed by Navier[2], who applied the equation of moments to the small flexure of a rod whose axis when unstrained is a parabola, and afterwards shewed[3] that when the initial form is a circle the equation can be integrated whether the displacement be small or not.

When a rod is bent so that the axis becomes a tortuous curve it is no longer possible to determine the form of the curve by taking moments about the normal to the osculating plane. The problem thus presented appears to have been first considered by Lagrange[4], who however fell into error on this point. A series of investigations by Binet, Bordoni, Poisson, Wantzel, and Saint-Venant gradually placed the matter in a clear light. Binet[5] introduced the equation of moments about the tangent; Poisson[6] arrived at the incorrect result that the torsional couple is constant from end to end of the rod; Wantzel[7] integrated the equations for a naturally straight wire of equal flexibility in all planes through its axis; and Saint-Venant[8] insisted on the importance of taking moments about the principal normal, and on the part played by the twist or angular displacement of the sections of the rod about the tangent. The simple fact that the equation of moments of the Bernoulli-Eulerian theory only applies to flexure in a principal plane appears to have been first noticed by Persy[9] in

[1] 'Genuina Principia...de statu æquilibrii et motu corporum...', *Nov. Comm. Acad. Petropolitanæ*, xv. 1771. The same assumption was made by Euler in 1744 (*Methodus inveniendi*..., p. 274).

[2] 'Sur la flexion des verges élastiques courbes', *Bulletin des Sciences par la Société Philomatique*, Paris, 1825. Navier does not appear to have been aware that Euler had anticipated him.

[3] *Résumé des Leçons...sur l'application de la Mécanique*, 1833.

[4] In his *Mécanique Analytique*, 1788.

[5] 'Mémoire sur l'expression analytique de l'élasticité...des courbes à double courbure', *Journal de l'École Polytechnique*, x. 1815.

[6] See Todhunter and Pearson's *History of the Elasticity and Strength of Materials*, vol. I. art. 433, and Poisson's *Mécanique*, t. I. pp. 622, sq. The error arises through supposing that there is a flexural couple about the binormal proportional to the curvature, which is true only when the section has kinetic symmetry.

[7] *Comptes Rendus*, XVIII. 1844.

[8] In papers in the *Comptes Rendus*, XVII. 1843, and XIX. 1844, reprinted in Saint-Venant's *Mémoires sur la Résistance des solides*..., Paris, 1844.

[9] Quoted by Saint-Venant in the *Historique Abrégé* prefixed to his edition of Navier's *Leçons*.

1834, and it led in the hands of Saint-Venant[1] to the theorem on asymmetric loading noticed in I., art. 108. It is noteworthy that Saint-Venant in 1844 gave the necessary equations for the determination of the curve formed by the axis of a thin rod or wire initially curved which is bent into a different tortuous curve. This was before he had made out the theories of flexure and torsion, and the equations he gave contain assumptions as to the flexural and torsional couples which could only be verified by some theory founded on the general equations of Elasticity. By means of his equations he solved some problems relating to circular wires and spiral springs.

The longitudinal and torsional vibrations of thin rods appear to have been investigated first experimentally by Chladni and Savart. The differential equation of longitudinal vibrations was discussed by Navier[2], and that of torsional vibrations by Poisson. The latter made a comparison of the frequencies of lateral, longitudinal, and torsional vibrations.

Some of the researches already mentioned were made after the discovery of the general equations, but, with the exception of Poisson's investigations of torsional vibrations, they do not rest upon an application of these equations. In his classical memoir of 1828[3] (one of the memoirs we have referred to in I. in connexion with the establishment of the general theory) there was given among the applications a general investigation of the equations of equilibrium and vibration of an elastic rod. The rod is regarded as a circular cylinder of small section, and the method employed is expansion of all the quantities that occur in terms of the distance of a particle from the axis of the cylinder. When terms above a certain order (the fourth power of the diameter) are rejected, the equations for flexural vibrations are identical with those already obtained by special hypotheses, as in Euler's theory of the vibrating rod, the equation obtained for extensional (longitudinal) vibrations is identical with that discussed by Navier. The equation of torsional vibrations appears to have been given here for the first

[1] *Comptes Rendus*, XVII. 1843, p. 1023.

[2] 'Solution de diverses questions relatives aux mouvements de vibration des corps solides', *Bulletin...Philomatique*, 1825. Chladni and Young appear to have been acquainted with the acoustics of longitudinal vibrations.

[3] 'Mémoire sur l'équilibre et le mouvement des corps élastiques', *Mém. Paris Acad.* VIII. 1829.

time. The chief point of novelty in the results is that the coefficients on which the frequencies depend are expressed in terms of the constants occurring in the general equations, but it was an immense advance in method when it was shewn that the equations generally admitted could be deduced from the mechanical theory. In his *Mécanique*[1], however, Poisson appears to have contented himself with the special hypotheses of his predecessors.

In the interval between this investigation of Poisson's and Kirchhoff's great memoir of 1858[2] the theory of thin rods was advanced by Wantzel and Saint-Venant in the manner already noticed, but in the same interval the researches of the latter on the torsion and flexure of prisms were opening a new path to discovery. It must always be remembered that these theories of Saint-Venant's are exact, not approximate. He gave the precise expressions for the displacements within a prismatic body to which forces are applied in a particular manner, and passed, by means of his "principle of the elastic equivalence of statically equipollent loads", to the conclusion that in a long thin prism the resultants only, and not the tractions that compose them, are significant. We may say that he arrived at certain "modes of equilibrium", and then shewed that they are the most important ones. His method was the *semi-inverse* method which we have described in vol. I.

Kirchhoff's method is in strong contrast both to Poisson's and to Saint-Venant's. He did not assume series and determine their coefficients, he found the necessary terms of the series with their coefficients; he did not make assumptions as to the displacements or the stresses, he found expressions for the displacements and the stresses; he never pretended to be exact, only to give a sufficient approximation. He formulated for the first time the proper way of applying the general equations to bodies whose linear dimensions are of different orders of magnitude. In any such body the relative displacements may be finite, while the strains are infinitesimal. The equations must be applied, not to the body as a whole, but to a small part of it whose dimensions are all of the same order of magnitude; and for such a part the equations admit of a

[1] The second edition is of date 1833. Poisson refers to his memoir for a more exact account.

[2] 'Ueber das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes', Crelle, LVI. 1859. See also Kirchhoff's *Vorlesungen über mathematische Physik, Mechanik*.

considerable simplification, viz.: for a first approximation bodily forces and kinetic reactions against acceleration may be neglected.

The process by which Kirchhoff developed his theory is largely kinematical. Suppose a thin rod is bent and twisted; every element of it will be bent and twisted more or less like one of Saint-Venant's prisms, but neighbouring elements must continue to fit—after the deformation there is still a continuous rod. To express this certain conditions are necessary, and they take the form of differential equations connecting the relative displacements of points in an element with the coordinates of their unstrained positions referred to axes fixed in the element, and with the position of the element in reference to the rod as a whole. Kirchhoff did not state the geometrical reason why his differential equations hold good[1]; he came upon them analytically and made them the basis of his discussion.

The next part of the work is approximate. The differential equations determine to some extent the character of the strain; replacing them by simpler equations, retaining only the most important terms, Kirchhoff was able to obtain a general account of the displacements, strains, and stresses within an element of the rod. In particular Saint-Venant's stress-conditions (I. arts. 82, 83) were shewn to hold to the order of approximation to which the work was carried. But Kirchhoff did not dwell upon this. He transformed at once the expression for the potential energy of the strained rod per unit of volume into an expression for the potential energy per unit of length, which he expressed as a quadratic function of the quantities that define the extension, the curvature, and the twist.

The discussion of the continuity and internal strain of the elements of the rod led to this expression for the potential energy, the general equations of equilibrium and small motion were deduced thence by an application of the principle of virtual work. When terminal forces only are applied, the equations, with certain interpretations of symbols, are identical with the equations of motion of a heavy rigid body about a fixed point. This is the celebrated theorem known as Kirchhoff's 'kinetic analogue'. He applied it to the discussion of a rod bent into a helix, and he

[1] The interpretation was given by Clebsch in his treatise of 1864, and the differential equations in question were obtained by M. Boussinesq in his memoir of 1871 by expressing the conditions of continuity.

proceeded to indicate an extension of his theory to the case of rods naturally curved. The application of the theorem of the kinetic analogue to integrate the equations of equilibrium of a thin rod has been considered very fully by Herr Hess[1] of Münich, and the theorem itself has been extended by Mr Larmor[2] to the case of a rod whose initial form is a helix.

The theory given by Clebsch[3] was founded partly on that of Kirchhoff, and partly on Saint-Venant's results for torsion and flexure. From the former he adopted the division of the rod into elements, and the conclusion that the internal strain in an element may be determined without reference to the bodily forces. He proceeded to explain that certain modes of equilibrium of a prism under terminal tractions were known, and from the results the resultant stresses and stress-couples at any section could be calculated. He then obtained the equations of equilibrium and small motion by the ordinary method of resolving and taking moments without invoking the principle of virtual work. Clebsch did valuable work in explaining the meaning of much of Kirchhoff's analysis, but his abandonment of the kinematical method was a retrograde step, and his adoption of Saint-Venant's stress-conditions as a basis of investigation laid his method open to criticism[4].

The *Natural Philosophy*[5] of Lord Kelvin and Professor Tait contained a new theory of thin rods. In this work there is a complete exposition of the kinematics of a curved rod, including the improvements introduced by Saint-Venant in his extensions of the Bernoulli-Eulerian theory. The values of the stress-couples are not deduced from the general theory of Elasticity, but from Kirchhoff's form of the potential energy, and this form it is attempted to establish by general reasoning. Many interesting applications are given, and in particular the theory of spiral springs is fully discussed.

In 1871 M. Boussinesq[6] came forward to criticise and interpret

[1] *Mathematische Annalen*, XXIII. 1884, and XXV. 1885.

[2] 'On the Direct application of the Principle of Least Action...', *Proc. Lond. Math. Soc.* XV. 1884.

[3] *Theorie der Elasticität fester Körper*, 1864.

[4] See for example Mr Basset's 'Theory of Elastic Wires', *Proc. Lond. Math. Soc.* XXIII. 1892.

[5] The date of the first edition is 1867.

[6] 'Étude nouvelle sur l'équilibre...des solides...dont certaines dimensions sont très petites...', Liouville's *Journal*, XVI. 1871.

Kirchhoff's theory. The fundamental idea of his researches is that the section of the rod is small, and that a solution of the problem to be useful must be approximate. By general reasoning, where Kirchhoff had employed pure analysis, he found Kirchhoff's expression for the extension of any fibre of the rod, and his differential equations of interior equilibrium of an element, and he gave an original proof that Saint-Venant's stress-conditions hold to the same order of approximation. M. Boussinesq's work is more general than that of Kirchhoff and Clebsch inasmuch as the æolotropy of his rod is limited only by the condition of symmetry of contexture with respect to the normal sections; it is less general inasmuch as the equations of equilibrium are applied to the rod as a whole, and thus the rod must be infinitely little bent and twisted. In a second memoir[1] M. Boussinesq has attempted to make his theory more rigorous by making the assumptions of his general reasoning more agnostic. The point on which he concentrates his objections to Kirchhoff's method is a step in the process whereby the exact differential equations expressing the continuity after deformation are replaced by approximate differential equations from which the character of the strain can be deduced. I think the objection can be removed. (See below ch. XV.)

Quite recently a new theory has been propounded by Mr Basset[2] with special reference to the case of a rod initially curved, of uniform circular section, of isotropic material, and very slightly deformed. He proposes to use the method, which Poisson used long ago for a straight rod, of expansion in powers of the distance from the elastic central-line. The analysis is very intricate, and the expressions found for the flexural and torsional couples do not agree with those given by Clebsch. Mr Basset appears to have been led to this subject by his researches on thin shells, and by dissatisfaction with Saint-Venant's stress-conditions as a basis for the theory of thin rods. We have already seen that in the theories of Kirchhoff and M. Boussinesq these conditions are not the foundation, but an incidental deduction, true to the order of approximation to which it is necessary to carry the work when the stress-couples only are required. Mr Basset's method has the advantage that the results are expressed directly in terms of the

[1] 'Complément à une étude de 1871...', Liouville's *Journal*, v. 1879.

[2] *Proc. Lond. Math. Soc.* XXIII. 1892.

displacements of a point on the elastic central-line in the directions of the tangent, principal normal, and binormal. In this, however, he was anticipated by Mr J. H. Michell[1], whose method also is more direct.

We have placed our accounts of the various theories of thin rods together because they exhibit a natural historical development, but before passing on to consider the applications it may be well to state the general character of the results. We have seen that an erroneous theory of the character of the strain led to an expression for the flexural couple, and that by corrections and modifications there sprang up the notion of two flexural couples and a torsional couple exerted about the principal axes of the cross-section and the tangent to the elastic central-line. Saint-Venant's theory of torsion gave the form of the torsional couple, and Clebsch shewed that the form of the flexural couples was actually that obtained in the older theories. The torsional couple is the product of the amount of the twist, (a quantity first properly expressed by Saint-Venant,) and a quantity called the torsional rigidity, which, in the case of an isotropic rod of uniform circular section, is the product of the modulus of rigidity and the moment of inertia of the section about an axis through its centroid perpendicular to its plane, but in other cases has a more complicated expression. The components of the flexural couple about the principal axes of inertia of the cross-section are the products respectively of the components of curvature of the elastic central-line about the same axes, and one of two quantities called the principal flexural rigidities, each of which is the product of the Young's modulus of the material for pull in the direction of the elastic central-line and the moment of inertia of the cross-section about a principal axis at its centroid. When the flexural and torsional couples are known the resultant stress across a normal section can be determined from the equations of equilibrium of an element bounded by two normal sections near together[2].

The applications of the theory of thin rods are very numerous. Prominent among them are the theory of the flexure of a piece by continuous load, and the theory of a continuous beam resting on supports.

[1] *Messenger of Mathematics*, XIX. 1890.

[2] This point was noticed by Kirchhoff in his memoir of 1858, but he did not utilise it to obtain the general equations.

The first of these problems was discussed by Navier[1] on the supposition of the Bernoulli-Eulerian theory that the flexural couple is proportional to the curvature, and the work of Clebsch shewed that this method is sufficiently correct when the length of the piece is great compared with its diameter, but it is a matter of interest to seek a solution of the problem which shall be exact instead of approximate. Investigations with this view have been made by Prof. L. Pochhammer and Prof. Pearson[2]. The treatment by the latter of the problem of a circular cylinder slightly bent by surface-tractions and by its own weight has been referred to in I. p. 34. The general result obtained is that when the length of the piece is as much as ten times its diameter Navier's formula may be safely used.

Prof. Pochhammer has given two investigations of the question of the circular cylinder deformed by surface-tractions. He considered[3] in the first place solutions of the equations of equilibrium periodic in the coordinate which gives the distance of a normal section from a fixed point on the axis of the cylinder. The solutions obtained are expressed in terms of Bessel's functions of the distance of a point from the axis of the cylinder. These solutions are exact but they may be converted into approximate solutions by expanding the Bessel's functions in series. When the ratio of the diameter to the length is regarded as small, and the surface-tractions are such as produce bending, it is found that Navier's formulæ are approximately verified. The same writer has also discussed the problem from a different point of view[4]. If we look at any such formula as those in I. p. 188, equations (119), we shall see that all the terms contributed to the displacements by Saint-Venant's solutions contain a function of the third degree in the coordinates of a point on the cross-section and the inverse fourth power of some quantity proportional to the linear dimensions of the section. Such terms are of negative degree in respect of quantities of the order of this linear dimension. Now by selecting the terms of various degrees in the general solutions which can be obtained Prof. Pochhammer

[1] In his *Leçons...sur l'application de la Mécanique...*, 1833.

[2] *Quarterly Journal*, XXIV. 1890.

[3] 'Beitrag zur Theorie der Biegung des Kreiscylinders', Crelle-Borchardt, LXXXI. 1876.

[4] *Untersuchungen über das Gleichgewicht des elastischen Stabes.* Kiel, 1879.

shewed that the terms of negative degree are correctly given by Navier's formulæ, *i.e.* they depend only on the resultant load, while the actual distribution of traction that produces the resultant load affects only terms of zero and positive degree. His work therefore determines the order of approximation to which the formulæ in question apply. In confirmation of these conclusions we may cite the work of Mr Chree[1] who has arrived at solutions of the same general character as those given by Prof. Pochhammer in his memoir of 1875[2].

The theory of a continuous beam resting on supports was at first very difficult, as a solution had to be obtained for each span by Navier's method, and the solutions compared in order to determine the constants. The analytical difficulties were very much reduced when Clapeyron[3] noticed that the flexural couples at three consecutive supports are connected by an invariable relation. This relation has been generalised by various writers, including Bresse[4], Herr Weyrauch[5], Prof. Pearson[6], and Mr Webb[7], but the analytical difficulties of any particular case of varying section and discontinuous load are still formidable. A method of graphical solution has however been invented by Mohr[8], and extended by Culmann[9], and it has to a great extent superseded the calculations from Clapeyron's Theorem of Three Moments.

This problem of the continuous beam is only one of the applications of the theory of thin rods to structures. Many applications to problems of frameworks will be found in such books as Müller-Breslau's *Die neueren Methoden der Festigkeitslehre* (Leipzig, 1886),

[1] 'On the Equations of an Isotropic Elastic Solid...', *Camb. Phil. Soc. Trans.* XIV. 1889.

[2] Crelle-Borchardt, LXXXI. 1876.

[3] *Comptes Rendus*, XLV. 1857, p. 1076. The history of Clapeyron's theorem is given by Mr J. M. Heppel in *Proc. R. S. Lond.* XIX. 1869.

[4] *Cours de Mécanique Appliquée*, t. III. Paris, 1865.

[5] Schlömilch's *Zeitschrift*, XVIII. XIX. 1873—4. See also Weyrauch's *Aufgaben zur Theorie elastischer Körper*, Leipzig, 1885.

[6] *Messenger of Mathematics*, XIX. 1890.

[7] *Proc. Camb. Phil. Soc.* VI. 1886.

[8] 'Beitrag zur Theorie des Fachwerks', *Zeitschrift des Architekten- und Ingenieur-Vereins zu Hannover*, 1874. This is the reference given by Müller-Breslau, but I have not seen the work. Lévy gives an account of the method in his *Statique Graphique*, t. II., and attributes it to Mohr. A slightly different account is given by Canevazzi in *Memorie dell' Accademia di Bologna* (4), I. 1880.

[9] *Die graphische Statik*, Bd. I. Zürich, 1875. See also Ritter, *Die elastische Linie und ihre Anwendung auf den continuirlichen Balken*. Zürich, 1883.

Weyrauch's *Theorie elastischer Körper* (Leipzig, 1884), and Ritter's *Anwendungen der graphischen Statik*, (Zürich, 1888). An immense literature has recently sprung up in this subject, but the use made of elastic theory is small.

The theory of the vibrations of thin rods deduced by Euler and Daniel Bernoulli from a special hypothesis is verified as an approximate theory by an application of Kirchhoff's results. Here also Prof. Pochhammer has done useful work by seeking exact solutions. He investigated[1] the modes of vibration of a circular cylinder which are periodic in respect of the coordinate measured along the axis, and he found that the simplest modes could be fairly described as longitudinal, torsional, and lateral. The expressions for the displacements involve Bessel's functions with arguments proportional to the distance from the axis, and, when these are expanded in series and the lowest powers retained, the frequency-equations are identical with those given by the ordinary theory. When higher powers are retained corrections are found to the frequency, and it appears that the velocities of longitudinal waves are only approximately independent of the wave-lengths. This correction has been obtained independently by Mr Chree[2], who has given an exhaustive account of longitudinal vibrations in a cylinder of finite thickness.

The vibrations of a curved bar whose initial form is a circle have been investigated by Prof. Lamb[3], Mr Michell[4], and Mr Basset[5]. The modes of flexural vibration of a complete circular ring which vibrates in its plane were discussed much earlier by Prof. Hoppe[6]. When there is flexure perpendicular to the plane of the ring there must be a comparable torsion, but the pitch is very nearly the same as in the corresponding mode involving flexure in the plane of the ring. There are also extensional and torsional modes of high pitch.

The principal problem arising out of the theory of longitudinal vibrations is the problem of impact. When two bodies impinge one of the effects produced is that each body is thrown into a state

[1] 'Ueber die Fortpflanzungsgeschwindigkeit kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder', Crelle-Borchardt, LXXXI. 1876.

[2] *Quarterly Journal*, XXI. 1886, XXIII. 1889, and XXIV. 1890.

[3] *Proc. Lond. Math. Soc.* XIX. 1888.

[4] *Messenger of Mathematics*, XIX. 1890.

[5] *Proc. Lond. Math. Soc.* XXIII. 1892.

[6] 'Vibrationen eines Ringes in seiner Ebene', Crelle-Borchardt, LXXIII. 1871.

of internal vibration, and it appears to have been hoped that a solution of the problem of the vibrations set up in two bars which impinge longitudinally would lead to an explanation of the laws of impact. Poisson[1] was the first to attempt a solution of the problem from this point of view. His method of integration in trigonometric series vastly increases the difficulty of deducing general results, and by an unfortunate error in the analysis he arrived at the paradoxical conclusion that when the bars are of the same material and section they never separate unless they are equal in length. Saint-Venant[2] treated the problem by means of discontinuous functions and arrived at certain results of which the most important relate to the duration of impact, and to the existence of an apparent "coefficient of restitution" for perfectly elastic bodies[3]. This theory is not confirmed by experiment. A correction suggested by Prof. Voigt[4], when worked out, led to little better agreement, and it appears that the attempt to trace the phenomena of impact to vibrations must be abandoned. Much more successful was the theory of Prof. Hertz[5]. He suggested that the phenomena may depend on a local statical effect gradually produced and gradually subsiding. The problem becomes a particular case of that solved by M. Boussinesq and Signor Cerruti which we have discussed in I. ch. IX. Prof. Hertz made an independent investigation of the particular case required, and found means of determining the duration of impact, and the shape and size of the parts of the surfaces of the impinging bodies that come into contact. His theory yielded a satisfactory comparison with experiment.

The theory of vibrations is capable of application to problems of resilience, or of the determination of the greatest strain produced in a body by forces suddenly applied. The particular problem of the longitudinal impact of a massive body upon one end of a rod was discussed by MM. Sébert and Hugoniot[6] and by M. Boussinesq[7]. The conclusions arrived at were tabulated and

[1] In his *Traité de Mécanique*, 1833.

[2] 'Sur le choc longitudinal de deux barres élastiques...', Liouville's *Journal*, XII. 1867.

[3] Cf. Hopkinson, *Messenger of Mathematics*, IV. 1875.

[4] Wiedemann's *Annalen*, XIX. 1882. See also Hausmaninger in Wiedemann's *Annalen*, XXV. 1885.

[5] 'Ueber die Berührung fester elastischer Körper'. Crelle-Borchardt, XCII. 1882.

[6] *Comptes Rendus*, XCV. 1882.

[7] *Applications des Potentiels...*, Paris, 1885. The results were given in a note in the *Comptes Rendus*, XCVII. 1883.

graphically illustrated by Saint-Venant[1]. But problems of resilience under impulses that tend to produce flexure are perhaps practically of more importance. When a body strikes a rod perpendicularly the rod will be thrown into vibration, and, if the body move with the rod, the ordinary solution in terms of the normal functions for the vibrations of the rod becomes inapplicable. Solutions of several problems of this kind in terms of the normal functions for the system consisting of the rod and the striking body were given by Saint-Venant[2].

Among problems of resilience we must note especially Willis's problem of the travelling load. When a train crosses a bridge the strain is not identical with the statical strain when the same train is standing on the bridge. To illustrate the problem thus presented Willis[3] proposed to consider the bridge as a straight wire and the train as a heavy particle deflecting it. Neglecting the inertia of the wire he obtained a certain differential equation which was subsequently solved by Sir G. Stokes[4]. Later writers have shewn that the effects of the neglected inertia are very important. The solution has been finally obtained by M. Phillips[5] and Saint-Venant[6], and an admirable *précis* of their results may be read in Prof. Pearson's *Elastical Researches of Barré de Saint-Venant.*

The success of theories of thin rods founded on special hypotheses appears to have given rise to hopes that a theory might be developed in the same way for thin plates and shells, so that the modes of vibration of a bell might be deduced from its form and the manner in which it is supposed to be supported. The first to attack this problem was Euler. In a note "De Sono Campanarum"[7] he proposed to regard the bell as divided into thin annuli each of which behaves as a curved bar. This leaves out of account the change of curvature in sections through the axis of

[1] In papers in *Comptes Rendus*, XCVII. 1883, reprinted as an appendix to his translation of Clebsch's *Theorie der Elasticität fester Körper.*

[2] In the 'Annotated Clebsch' just referred to, *Note du* § 61.

[3] In an appendix to the *Report of the Commissioners appointed to enquire into the application of Iron to Railway Structures*, 1849.

[4] 'Discussion of a Differential Equation relating to the Breaking of Railway Bridges', *Camb. Phil. Soc. Trans.* VIII. 1849. (See also Sir G. Stokes's *Math. and Phys. Papers*, vol. II.)

[5] 'Calcul de la résistance des poutres droites...sous l'action d'une charge en mouvement', *Ann. des Mines*, VII. 1855.

[6] In the 'Annotated Clebsch', *Note du* § 61.

[7] *Novi Commentarii...Petropolitanæ*, x. 1764.

the bell. James Bernoulli[1] (the younger) followed. He assumed the shell to consist of a kind of double sheet of curved bars, the bars in one sheet being perpendicular to those in the other. Reducing the shell to a plane plate he found an equation of vibration which we now know is incorrect.

James Bernoulli's attempt appears to have been made with the view of discovering a theoretical basis for the experimental results of Chladni concerning the nodal figures of vibrating plates. These results were still unexplained when in 1809 the French *Institut* proposed as a subject for a prize the investigation of the tones of a vibrating plate. After several attempts the prize was adjudged in 1815 to Mdlle. Sophie Germain, and her work was published in 1821[2]. She assumed that the sum of the principal curvatures of the plate when bent plays the same part in the theory of plates as the curvature of the elastic central-line does in the theory of rods, and she proposed to regard the work done in bending as proportional to the integral of the square of the sum of the principal curvatures taken over the surface. From this assumption and the principle of virtual work she deduced the equation of flexural vibration in the form now generally admitted. Later investigations have shewn that the form assumed for the work done in bending is incorrect. Navier[3] followed Mdlle. Germain in her assumption, obtained practically the same differential equation, and applied it to the solution of some problems of equilibrium.

After the discovery of the general equations of Elasticity little advance seems to have been made in the treatment of the problem of shells for many years, but the more special problem of plates attracted much attention. Poisson[4] and Cauchy[5] both treated the latter, proceeding from the general equations of Elasticity, and supposing that all the quantities which occur can be expanded

[1] 'Essai théorique sur les vibrations des plaques élastiques...', *Nova Acta... Petropolitanæ*, v. 1787.

[2] *Recherches sur la théorie des surfaces élastiques*. Paris, 1821.

[3] 'Extrait des recherches sur la flexion des plans élastiques', *Bulletin... Philomatique*, 1823.

[4] In the memoir of 1828. A large part of the investigation is reproduced in Todhunter and Pearson's *History of the Elasticity and Strength of Materials*, vol. I.

[5] In an article 'Sur l'équilibre et le mouvement d'une plaque solide' in the *Exercices de Mathématiques*, vol. III. 1828. Most of this article also is reproduced by Todhunter and Pearson.

in powers of the distance from the middle-surface. The equations of equilibrium and free vibration when the displacement is purely normal were deduced. Much controversy has arisen concerning Poisson's boundary-conditions. These expressed that the resultant forces and couples applied at the edge must be equal to the forces and couples arising from the strain. In a famous memoir[1] Kirchhoff shewed that these conditions express too much and cannot in general be satisfied. His method rests on two assumptions, (1) that line-elements of the plate initially normal to the middle-surface remain normal to the middle-surface after strain, and (2) that all the elements of the middle-surface remain unstretched. These assumptions enabled him to express the potential energy of the bent plate in terms of the curvatures produced in its middle-surface. The equations and conditions were then deduced by the principle of virtual work, and applied to the problem of the flexural vibrations of a circular plate.

The theory of plates comes under the general theory (referred to above) of bodies some of whose dimensions are infinitely small in comparison with others. The application of Kirchhoff's method of treating such bodies to the problem was made by Gehring[2], a pupil of Kirchhoff's, at the suggestion of the latter. The method is precisely similar to that adopted by Kirchhoff in the case of thin rods. The conditions of continuity of the plate when deformed are expressed by certain differential equations connecting the relative displacements of a point within an element with the unstrained coordinates of the point referred to axes fixed in the element, and with the position of the element in reference to the plate as a whole. These equations are replaced by approximate ones retaining the most important terms, and from these a general approximate account of the displacement, strain, and stress within an element is deduced. To the order of approximation to which the work is carried there is no stress across planes parallel to the middle-surface. An expression for the potential energy is obtained. This expression consists of two parts, one a quadratic function of

[1] 'Ueber das Gleichgewicht und die Bewegung einer elastischen Scheibe'. Crelle's *Journal*, XL. 1850.

[2] 'De Æquationibus differentialibus quibus æquilibrium et motus laminæ crystallinæ definiuntur'. Berlin, 1860. The analysis may be read in Kirchhoff's *Vorlesungen über mathematische Physik, Mechanik*, and some part of it also in Clebsch's *Theorie der Elasticität fester Körper*.

the quantities defining the extension of the middle-surface with a coefficient proportional to the thickness of the plate, and the other a quadratic function of the quantities defining the flexure of the middle-surface with a coefficient proportional to the cube of the thickness. The equations of small motion are deduced by an application of the principle of virtual work. When the displacement of a point on the middle-surface is infinitesimal the flexure depends only on normal displacements, and the extension only on tangential displacements, and the equations divide into two sets. The equation of normal vibration and the boundary-conditions are those previously found and discussed by Kirchhoff[1].

In his *Theorie der Elasticität fester Körper* Clebsch gave a much more detailed investigation of the theory. He adopted Gehring's method as far as the discussion of the interior equilibrium of an element, with the result that to a certain order of approximation there is no stress across planes parallel to the middle-surface. After a geometrical discussion of the developable into which the middle-surface is transformed by bending, he proceeded to form the general equations by aid of the principle of virtual work. In this process he introduces a number of arbitrary multipliers, as the variation has to be made subject to certain geometrical conditions. The work is very intricate but I think it contains a flaw. The equations are ultimately transformed into a shape in which they represent the statical conditions of equilibrium of an element bounded by plane faces perpendicular to the middle-surface. Now Clebsch has retained the supposition that the stress-components parallel to lines on the middle-surface exerted across plane-elements parallel to the same surface vanish, and it follows that the stress-resultants normal to the middle-surface exerted across planes which are also normal to this surface vanish. The latter stress-resultants are known not to vanish, but to be of the same order as the stress-couples across the same faces[2]. This is the point on which I conceive that Clebsch has fallen into error, and I cannot find that the arbitrary multipliers which he has introduced enable him to evade the difficulty.

The equations for a plate slightly bent in terms of stress-resultants and stress-couples appear to have been first correctly

[1] In his memoir of 1850. See also *Vorlesungen über mathematische Physik, Mechanik.*

[2] See below ch. xx.

given in the *Natural Philosophy* of Lord Kelvin and Professor Tait. These authors make little use of the theory of Elasticity, but deduce the values of the stress-couples by a method similar to that which they employed in the case of thin rods. More recent expositions of the same theory have been given by Saint-Venant[1], and by Mr Basset[2] and Prof. Lamb[3]. The two latter were led thereto by the difficulty in the subject of thin shells to be referred to presently.

We have already noticed that the boundary-conditions obtained by Kirchhoff differ from those found by Poisson. The union of two of Poisson's boundary-conditions in one of Kirchhoff's was first explained in the *Natural Philosophy* above referred to. It was remarked that the couples acting on any element of the edge can be resolved into tangential and normal components; of these the latter may be replaced by pairs of forces normal to the middle-surface, and the difference of these forces in consecutive elements gives rise to a resultant force in the direction of this normal. Thus there is no equation of couples about the normal to the edge.

M. Boussinesq[4] has applied to the theory of plates a method similar to that by which he treated the theory of rods. He shewed in a very elementary manner from the equations of Elasticity that the stress exerted across any section of the plate parallel to its middle-surface must be small compared with that which is exerted across a section in any other direction. Taking the equations thus derived as a basis of approximation, he deduced equations and boundary-conditions which agree with Kirchhoff's. The method of considering the boundary-conditions invented by Lord Kelvin and Professor Tait was rediscovered by M. Boussinesq. In a later memoir[5] the same author has proposed to modify his theory by assuming only that the stresses are slowly varying from point to point of the middle-surface.

Lord Rayleigh[6] has gathered up the threads of the theory so far as it relates to vibrations. Starting from Kirchhoff's expression for the potential energy of strain, he proceeded to deduce, by the method of virtual work, the differential equations and boundary-conditions for free vibrations. He followed Kirchhoff in the discussion of the circular plate, and gave the solutions of the

[1] In the 'Annotated Clebsch', *Note du* § 73.
[2] *Proc. Lond. Math. Soc.* XXI. 1890.
[3] *Ibid.*
[4] Liouville's *Journal*, XVI. 1871.
[5] *Ibid.* V. 1879.
[6] *Theory of Sound*, vol. I. ch. X.

frequency-equation for the lowest tones and the character of the nodal lines. He remarked that if there are any materials which are not laterally contracted when they are slightly extended longitudinally, rectangular plates of such materials can vibrate like bars without being supported at the edges. The theory of the vibrations of a free rectangular plate has not been made out except in these cases.

The first attempt to solve the problem of thin curved plates or shells by the aid of the general theory of Elasticity was made by Herr Aron[1]. He expressed the geometry of the middle-surface by means of two parameters after the manner of Gauss, and he used the method employed by Clebsch for plates to obtain the equations of equilibrium and small motion. The equations found are very complicated, and little result can be obtained from them. It is however to be noted that the expression for the potential energy is of the same form as in the case of the plane plate, the quantities defining the curvature of the middle-surface being replaced by the differences of their values before and after strain.

M. Mathieu[2] adapted the method of Poisson for plane plates to the case where the middle-surface is of revolution, and the deformation very small. He noticed that the modes of vibration possible to a shell are not separable into normal and tangential modes as in the case of a plane plate, and the equations that he found are those that would be deduced from Herr Aron's form of the potential energy if only the terms depending on the stretching of the middle-surface were retained. For a spherical shell with a circular edge he obtained the solution of his equations in terms of functions which are really generalised spherical harmonics.

Lord Rayleigh[3] attacked the problem of open spherical shells from a different point of view. He concluded from physical reasoning that in a vibrating shell the middle-surface remains unstretched. When this is the case the component displacements must have certain forms. He assumed that the potential energy is

[1] 'Das Gleichgewicht und die Bewegung einer unendlich dünnen beliebig gekrümmten elastischen Schale'. Crelle-Borchardt, LXXVIII. 1874.

[2] 'Mémoire sur le mouvement vibratoire des Cloches'. *Journ. de l'École Polytechn.* LI. 1883.

[3] 'On the Infinitesimal Bending of Surfaces of Revolution'. *Proc. Lond. Math. Soc.* XIII. 1882.

a quadratic function of the changes of principal curvature. From this he easily deduced the form of the normal functions and the frequencies of the component tones.

By an adaptation of Gehring's theory of plates I found[1] that Lord Rayleigh's solutions fail to satisfy the boundary-conditions which hold at the edges of a shell vibrating freely, and I proposed to adopt for bells M. Mathieu's theory that the extension is the governing circumstance. This raised a discussion in which Mr Basset and Prof. Lamb took part. Their researches, as well as later researches by Lord Rayleigh, have done much to elucidate the subject. It has been shewn to be probable that the extensional strain, proved to exist, is practically confined to a narrow region near the edge, and within that region is of such importance as to secure the satisfaction of the boundary-conditions, while the greater part of the shell vibrates according to Lord Rayleigh's type. For the developments of the theory the reader is referred to chapters XXI. and XXII. of this volume.

Whenever very thin rods or plates are employed in constructions it becomes necessary to consider whether the forces in action can cause such pieces to buckle. The flexibility of a long shaft held vertically must have been noticed by most observers, and we have already seen that Euler and Lagrange treated the subject. One of the forms of the *elastica* included in Euler's classification is a curve of sines of small amplitude, and Euler pointed out[2] that in this case the line of thrust coincides with the unstrained axis of the rod, so that the rod, if of sufficient length and vertical when unstrained, may be bent by a weight attached to its upper end. Further investigations[3] led him to assign the least length of a column in order that it may bend under its own or an applied weight. Lagrange[4] followed and used his theory to determine the strongest form of column. He found that the circular cylinder is among the forms of maximum efficiency, and for small departures from the cylindrical form it is actually the strongest form of column. These two writers found a certain length which a column must attain to be bent by its own or an applied weight, and they

[1] 'On the Small Free Vibrations and Deformation of a Thin Elastic Shell'. *Phil. Trans. R. S.* (A), 1888.

[2] *Hist. Berlin. Acad.* XIII. 1757.

[3] *Acta Acad. Petropolitanæ* for 1778, *Pars Prior*, pp. 121—193.

[4] *Misc. Taur.* v. 1773.

concluded that for shorter lengths it will be simply compressed while for greater lengths it will be bent.

The problem of Euler was the only problem in Elastic Stability attempted for many years. It found its way into practical treatises on the 'Resistance of Materials', as Heim's *Gleichgewicht und Bewegung gespannter elastischer fester Körper*[1] and Bresse's *Cours de Mécanique Appliquée*[2]. Meanwhile Lamarle[3] pointed out that, unless the length be very great compared with the diameter, a load great enough to produce buckling will be more than great enough to produce set. The problem was taken up again by Prof. Greenhill, who assigned a slightly different limit to Euler's for the height consistent with stability[4], and extended the theory to the case where the rod is twisted, and to the case where it rotates[5], with applications to the strength of screw-propeller shafts. Mr Chree[6] has recently shewn that, unless the length be very great compared with the diameter, there is a danger, at any rate in hollow shafts, of set being produced by a smaller velocity of rotation than that which Prof. Greenhill assigns for the commencement of buckling. He has done for Prof. Greenhill's problem of the rotating shaft what Lamarle did for Euler's problem of the column.

M. Lévy[7] and Halphen[8] have treated the case of a circular ring bent by uniform normal pressure. Unless the pressure exceed a certain limit the ring contracts radially, but when the limit is passed it bends under the pressure. M. Lévy appears to think that the result gives the limiting pressure consistent with stability for a circular cylindrical shell, but he has omitted to notice that the constant of flexural rigidity for a curved bar is quite different to that for a cylinder.

In all the researches that we have mentioned the methods employed have been tentative. Two modes of equilibrium have

[1] Stuttgart, 1838.

[2] *Première Partie.* Paris, 1859.

[3] 'Mémoire sur la flexion du bois'. *Annales des Travaux publics de Belgique*, IV. 1846.

[4] *Camb. Phil. Soc. Proc.* IV. 1881.

[5] *Proc. Inst. Mech. Engineers*, 1883.

[6] *Camb. Phil. Soc. Proc.* VII. 1892.

[7] 'Mémoire sur un nouveau cas intégrable du problème de l'élastique et l'une de ses applications'. Liouville's *Journal*, X. 1884.

[8] *Comptes Rendus*, XCVIII. 1884.

been known, one involving flexure, and the other involving simple compression or simple torsion, and it was concluded that in such cases whenever flexure is possible it will take place; but there existed no theory for determining *a priori* when two modes of equilibrium are possible, or why flexure should take place whenever it is a characteristic of one possible mode.

These defects were remedied by Mr G. H. Bryan[1]. In regard to the first point it had been shewn by Kirchhoff that when a body is held in equilibrium by given surface-tractions the state of strain produced is unique. The exceptions to which we have alluded are more apparent than real. Kirchhoff's proof depended on the variation of the energy, and he considered only first variations. In the cases we have mentioned a small displacement really changes the character of the surface-tractions, and it becomes necessary to consider second variations. Take for example the slender vertical column supporting a weight. When it is simply compressed the weight gives rise to purely longitudinal tractions. But now let the column be slightly displaced by bending; there will be a component of the weight tending to produce longitudinal traction, and a component tending to produce transverse traction; the surface-tractions are really changed in character by the displacement. For a short thick column the change is of the kind which it is legitimate to neglect, for a long thin one it becomes important.

Now Mr Bryan has shewn that there are only two cases of possible instability, (1) where nearly rigid-body displacements are possible with very small strains, as when a sphere is put into a ring which fits it tightly along a great circle, and (2) where one of the dimensions of a body is small in comparison with another, as in a thin rod or plate. He proceeded by taking the second variation of the energy-function, and he pointed out that, as in every case the system tends to take up the position in which the potential energy is least, modes involving flexure will be taken by a thin rod or plate under thrust whenever such modes are possible. Mr Bryan has given several interesting applications of his theory which we shall consider in our last chapter.

[1] *Camb. Phil. Soc. Proc.* VI. 1888.

CHAPTER XIII.

THE BENDING OF RODS IN ONE PLANE.

212. The "thin rod" of Mathematical Physics is an elongated body of cylindrical form. The sections of the rod perpendicular to the generating lines of its cylindrical bounding surface are called its *normal sections*, the line parallel to these generating lines which is the locus of the centroids of the normal sections is called the *elastic central-line* or *axis* of the rod. The principal axes of inertia of any normal section at its centroid and the elastic central-line are called the *principal torsion-flexure axes* of the rod at the point where the normal section cuts the elastic central-line, and the planes through these principal axes of inertia and the elastic central-line are called the *principal planes* of the rod.

When the rod is deformed the particles that initially were in the elastic central-line come to lie on a curved line which will be called the *strained elastic central-line*, and the particles that initially were in a normal section come to lie in general on a curved surface which is not truly normal to the strained elastic central-line. We may define the *principal planes* at any point of the elastic central-line of the strained rod to be two perpendicular planes through the tangent to the strained elastic central-line, of which one contains the tangent to the line of particles that initially coincided with one principal axis of inertia of the normal section; the principal torsion-flexure axes of the strained rod will be the tangent to the strained elastic central-line and the intersections of the normal section by the principal planes.

When forces in a principal plane are applied to the unstrained rod it can be proved by an application of the general theory of

Elasticity that the strained elastic central-line is a curved line lying in that plane. In the present chapter we shall for the most part confine our attention to the bending of the rod in a principal plane, which plane will be called the *plane of flexure.*

In case the normal sections of the rod have kinetic symmetry, *e.g.* when the section is a circle, a square, an equilateral triangle, the principal planes are indeterminate, and any plane through the elastic central-line is a principal plane.

213. Stress System.

Suppose that the rod is held bent in a principal plane by forces and couples suitably applied, and consider the stress across any normal section. Let P be any point on the strained elastic central-line, and suppose the normal section at P drawn. The actions of the particles on one side of this section upon the particles on the other side can be reduced to a force and couple at P. The force can be resolved into two components: T along the tangent to the strained elastic central-line at P, and N along the normal to the same line drawn inwards towards the centre of curvature. The axis of the couple is perpendicular to the plane of flexure. The couple will be denoted by G, and it will be called the *flexural couple* or *bending moment* at P. The forces T and N will be called respectively the *tension*, and the *shearing force* at P.

By an application of the general theory of Elasticity it can be shewn that the flexural couple G is proportional to the curvature of the strained elastic central-line at P, so that if ρ be the radius of curvature of this line,

$$G = \mathfrak{B}/\rho,$$

where $\mathfrak{B}$ is a constant depending on the nature of the material and the form of the section. This constant is called the *flexural rigidity* for the plane in question. Further it can be shewn that the constant $\mathfrak{B}$ is the product of the Young's modulus E of the material, and the moment of inertia I_1 of the normal section about that principal axis at its centroid which is perpendicular to the plane of flexure, so that

$$\mathfrak{B} = EI_1 = E\omega k_1^2,$$

where ω is the area of the normal section, and k_1 its radius of gyration about an axis through its centroid perpendicular to the plane of flexure.

When the initial form of the elastic central-line is a plane curve in a principal plane of the rod, and the strained form is a different plane curve in the same plane, the flexural couple at any point is proportional to the change of curvature of the element of the elastic central-line at the point, so that

$$G = \mathfrak{B}(1/\rho - 1/\rho_0),$$

where ρ_0 and ρ are the initial and final radii of curvature.

We postpone to ch. XV. the proofs of the propositions that depend on the general theory of Elasticity.

214. General Equations of Equilibrium.

The general equations of equilibrium of the rod, when held bent by forces X, Y per unit of length directed along the tangent and normal at any point, and a couple M per unit length about an axis at the same point perpendicular to the plane of flexure, can now be written down.

Let PP' be an element of the elastic central-line of length ds, and take as temporary axes of coordinates x, y the tangent and normal to the elastic central-line at P. Then the external forces applied to the element may be reduced to a force at P whose components are Xds, Yds parallel to the axes x, y, and

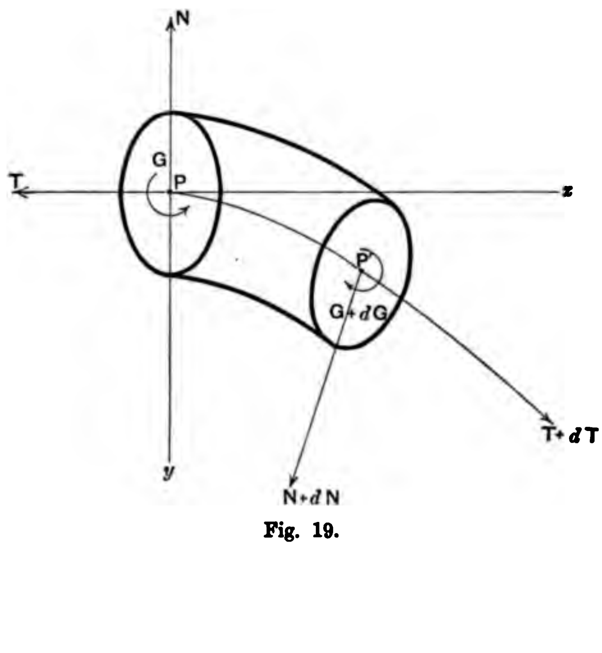


Fig. 19.

a couple Mds which tends to turn the element PP' from x towards y.

The forces that act on the element PP' arising from the action of neighbouring elements are shewn in the diagram (fig. 19). The part to the left of P gives us forces T, N, and a couple G, the part to the right of P' gives us forces $T+dT$, $N+dN$, and a couple $G+dG$, in the directions shewn.

If $d\phi$ be the angle which the tangent at P' to the elastic central-line makes with the tangent at P, these forces and couples reduce to a force at P whose components are

$$-T+(T+dT)\cos d\phi-(N+dN)\sin d\phi \text{ parallel to } x,$$

$$-N+(N+dN)\cos d\phi+(T+dT)\sin d\phi \text{ parallel to } y,$$

and a couple whose moment is

$$-G+(G+dG)+(N+dN)\,ds.$$

Adding together all the forces parallel to x, all the forces parallel to y, and all the couples, and equating the results severally to zero, we obtain the equations of equilibrium in the form

$$\left.\begin{aligned}\frac{dT}{ds}-\frac{N}{\rho}+X=0,\\ \frac{dN}{ds}+\frac{T}{\rho}+Y=0,\\ \frac{dG}{ds}+N+M=0\end{aligned}\right\}\quad\text{.......................(1)},$$

in which $\rho=ds/d\phi$ is the radius of curvature of the elastic central-line at P.

In addition to the equations of equilibrium (1) that hold at every point of the rod there are certain conditions to be satisfied at the extremities. When given forces and couples are applied at the extremities these conditions take the form of equations connecting the values of T, N, G at the extremities with the given component forces and couples. When an end is free T, N, and G have to vanish there. When an end is fixed the constraint which fixes it may be such as to allow of the extremity taking up different directions or it may be such that the direction is fixed. In the former case the end will be described as *simply supported*, in the latter as *built-in*.

215. Horizontal Rod very little bent.

We shall consider in the first place the very important case of a straight rod resting on rigid supports at the same level, and slightly bent by vertical forces supposed to act in a principal plane. In this case $1/\rho$ is always very small, and X everywhere vanishes, so that by the first of equations (1) the tension T need not be considered. We may therefore suppose the stress-system to reduce to the shearing force N and the couple G. In accordance with a remark already made we may take G to be proportional to the curvature $1/\rho$ of the elastic central-line, and, if the axes of x and y be taken horizontally and vertically, we may replace $1/\rho$ by $\pm d^2y/dx^2$, the sign being determined so that the sense of G may be that of a couple tending to increase the curvature.

With regard to the terminal conditions it is to be noticed that at a simply supported end G vanishes, so that d^2y/dx^2 vanishes while y has a given value. At a built-in end y and dy/dx have given values.

In problems of the kind we are now entering upon the fact that G is proportional to d^2y/dx^2, and the ordinary principles of Statics, are together sufficient to determine the form of the strained elastic central-line, and the pressures on the supports can be deduced. (Cf. I. art. 107.)

Since there is no applied couple M, the third of equations (1) becomes

$$\frac{dG}{dx} + N = 0,$$

where dx is written for ds since the strained elastic central-line very nearly coincides with its unstrained position. This equation gives the shearing force at any point.

216. Rod loaded at one end.

We shall first investigate the form of a rod initially straight which is loaded at one end with a weight W while the other end is built-in horizontal, the vertical plane through the unstrained elastic central-line being a principal plane of the rod.

Taking the origin at the built-in end, the axis y downwards, and the axis x horizontal, and writing y for the vertical displacement

at any point of the elastic central-line, the flexural couple is $\mathfrak{B}d^2y/dx^2$, and the equation of equilibrium found by taking moments about a point P for the part between P and the loaded end is

$$\mathfrak{B}\frac{d^2y}{dx^2} - W(l-x) = 0,$$

Fig. 20.

where x is the distance of P from the built-in end, and l is the length of the rod.

Since y and dy/dx both vanish when $x = 0$, this equation gives the deflexion y at any point in the form

$$\mathfrak{B}y = \tfrac{1}{2}Wx^2(l - \tfrac{1}{3}x).$$

The shearing force at any section is clearly equal to W.

When the vertical plane through the elastic central-line is not a principal plane of the rod, we may suppose that W is resolved into two components, one in each principal plane. There will be two constants of flexural rigidity B_1, B_2 proportional to the two principal moments of inertia I_1, I_2 of the cross-section at its centroid. The displacements y_1, y_2 in the two planes are given by the formulæ

$$B_1y_1 = \tfrac{1}{2}W\cos\theta\, x^2(l - \tfrac{1}{3}x),$$
$$B_2y_2 = \tfrac{1}{2}W\sin\theta\, x^2(l - \tfrac{1}{3}x),$$

where θ is the angle between the vertical plane through the elastic central-line and the principal plane in which the displacement is y_1.

We may regard y_1, y_2, x as Cartesian rectangular coordinates of a point on the strained elastic central-line and it then appears that this is a plane curve but not in a vertical plane. The plane in which it lies is given by the equation

$$y_2/y_1 = \tan\theta\, B_1/B_2.$$

If ϕ be the angle which the plane perpendicular to the plane of the curve makes with the principal plane in which the displacement is y_1, we have

$$\tan\phi\tan\theta = -B_2/B_1 = -I_2/I_1,$$

where I_1, I_2 are the two principal moments of inertia of the normal section at its centroid, so that this plane and the vertical plane cut the normal section in lines which are conjugate diameters of the ellipse of inertia. This is the theorem of Saint-Venant and Bresse given in I. art. 108.

In the remaining problems that will be here considered the plane of flexure will be assumed to be a principal plane of the rod.

217. Uniform Load, supported ends.

Suppose a long rod or beam of uniform section and material simply supported at its extremities, and bent by its own weight.

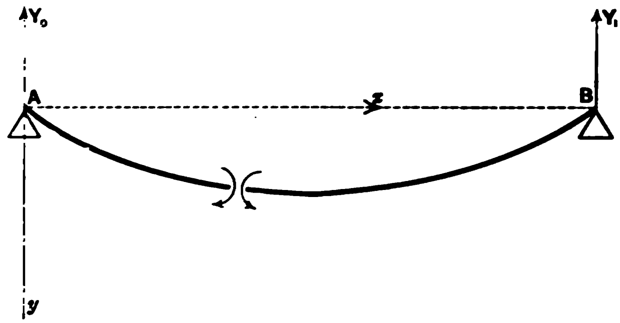


Fig. 21.

Let l be the length, $\mathfrak{B}$ the flexural rigidity, Y_0 and Y_1 the pressures on the ends, w the weight of the beam per unit of length.

Take the point of support A as origin, and the axes as in the figure.

We have the equation of equilibrium of the part between A and any point P by moments about P,

$$\mathfrak{B}\frac{d^2y}{dx^2} + Y_0x - \tfrac{1}{2}wx^2 = 0 \text{ .... (2)},$$

where the last term is $\int_0^x (x-\xi)\,wd\xi$ the moment of the weight of the part AP.

Since $d^2y/dx^2 = 0$ when $x = l$, the particular form of this equation when $x = l$ is

$$Y_0 l - \tfrac{1}{2} w l^2 = 0,$$

so that $Y_0 = \tfrac{1}{2} wl$ as is otherwise obvious.

We may integrate the above equation in the form

$$\mathfrak{B}(y - x \tan \alpha) + \tfrac{1}{6} Y_0 x^3 - \tfrac{1}{24} w x^4 = 0 \quad\ldots\ldots\ldots\ldots (3),$$

where α is the angle the axis at A makes with the horizontal; and since $y = 0$ when $x = l$

$$\mathfrak{B} l \tan \alpha = \tfrac{1}{6} Y_0 l^3 - \tfrac{1}{24} w l^4 = \tfrac{1}{24} w l^4 \quad\ldots\ldots\ldots\ldots\ldots (4).$$

Hence the deflexion y is given by the equation

$$\mathfrak{B} y = \tfrac{1}{24} w x (x^3 + l^3 - 2 l x^2) = \tfrac{1}{24} w x (l - x) [l^2 + l x - x^2] \ldots (5).$$

If we refer to the middle point of the beam as origin and write $x' + \tfrac{1}{2} l$ for x, we have

$$\mathfrak{B} y = \tfrac{1}{24} w (\tfrac{1}{4} l^2 - x'^2)(\tfrac{5}{4} l^2 - x'^2) \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

and the central deflexion is

$$y_0 = \tfrac{5}{384} w l^4 / \mathfrak{B} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

218. Uniform Load, built-in ends.

When the extremities are built-in, and horizontal, and the beam is bent by its own weight, let M be the bending moment at A or B, the rest of the notation being the same as before.

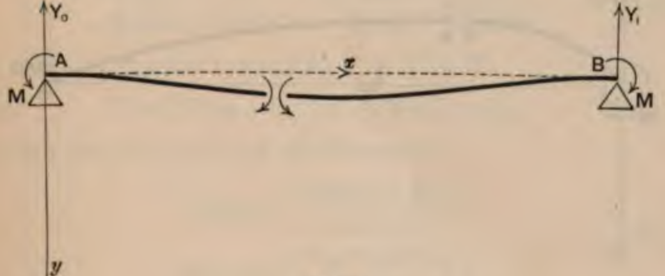


Fig. 22.

We have at once the equation of moments

$$\mathfrak{B} \frac{d^2 y}{dx^2} - M + Y_0 x - \tfrac{1}{2} w x^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots (8),$$

with the terminal conditions that y and dy/dx vanish when $x=0$ and when $x=l$. We have on integrating

$$\mathfrak{B}y = \tfrac{1}{2}Mx^2 - \tfrac{1}{6}Y_1x^3 + \tfrac{1}{24}wx^4 \text{ ....(9)},$$

where
$$M - \tfrac{1}{3}Y_1l + \tfrac{1}{12}wl^2 = 0,$$
$$M - \tfrac{1}{2}Y_1l + \tfrac{1}{6}wl^2 = 0:$$

from which
$$\left.\begin{aligned} M &= \tfrac{1}{12}wl^2, \\ Y_1 &= \tfrac{1}{2}wl \end{aligned}\right\} \text{ .... (10)}:$$

and as before, referring to the middle point as origin and writing $(x' + \frac{1}{2}l)$ for x, we find

$$\mathfrak{B}y = \tfrac{1}{24}w\left(\tfrac{1}{4}l^2 - x'^2\right)^2 \text{ ....(11)},$$

and the central deflexion is $\frac{1}{384}wl^4/\mathfrak{B}$ or $\frac{1}{5}$ of what it would be if the ends were simply supported.

The reader will find it easy to prove in like manner that for a beam of length l supported at its middle point the deflexion y at a distance x from the middle point is given by the equation

$$\mathfrak{B}y = \tfrac{1}{48}wx^2\left(3l^2 - 4lx + 2x^2\right),$$

so that the terminal deflexion is $\frac{1}{128}wl^4/\mathfrak{B}$ or $\frac{3}{5}$ of that at the middle point of the same beam when its ends are supported.

Another particular example will be found by taking a beam of length l without weight having one extremity simply supported and a given bending couple M being applied at the other. It will be found that the pressure on the support at the end at which M is applied is M/l, while the other extremity must be pressed in the

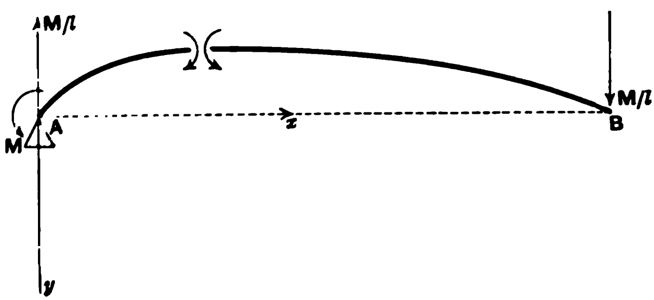


Fig. 23.

opposite direction with the same force. The equation of the central-line referred to the extremity where the couple is applied is

$$\mathfrak{B}y = -\tfrac{1}{6}Mx\,(l-x)\,(2l-x)/l.$$

219. Isolated load, supported ends.

Suppose the beam weightless and supporting a weight W at a point Q distant a from one end, and b from the other, where $a+b=l$, and suppose the ends simply supported.

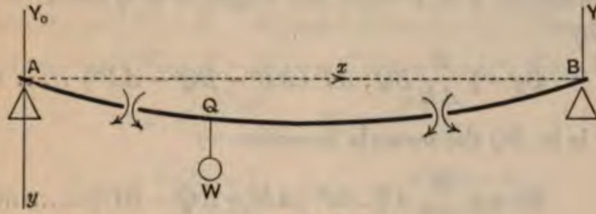


Fig. 24.

Let Y_0 and Y_1 be the pressures at A and B.

Then in AQ $$\mathfrak{B}\frac{d^2y}{dx^2}+Y_0x=0 \quad\text{......................}\quad (12);$$

and in BQ $$\mathfrak{B}\frac{d^2y}{dx^2}+Y_1(l-x)=0 \quad\text{..................}\quad (13).$$

We may integrate these in the forms

$$\left.\begin{aligned}\mathfrak{B}\,(y-x\tan\alpha)&=-\tfrac{1}{6}Y_0x^3,\\ \mathfrak{B}\,\{y-(l-x)\tan\beta\}&=-\tfrac{1}{6}Y_1(l-x)^3\end{aligned}\right\}\quad\text{.........}\quad(14).$$

Since y and $\dfrac{dy}{dx}$ are the same on either side of Q

$$\left.\begin{aligned}\mathfrak{B}a\tan\alpha-\tfrac{1}{6}Y_0a^3&=\mathfrak{B}b\tan\beta-\tfrac{1}{6}Y_1b^3,\\ \mathfrak{B}\tan\alpha-\tfrac{1}{2}Y_0a^2&=-\mathfrak{B}\tan\beta+\tfrac{1}{2}Y_1b^2\end{aligned}\right\}\text{.........}(15).$$

By ordinary Statics $Y_0a=Y_1b,\ Y_0+Y_1=W.$

Whence $\qquad Y_0=Wb/(a+b),\quad Y_1=Wa/(a+b).$

Solving (15) for $\tan\alpha$ and $\tan\beta$ we have

$$\left.\begin{aligned}\mathfrak{B}\tan\alpha&=\frac{Wab\,(a+2b)}{6\,(a+b)},\\ \mathfrak{B}\tan\beta&=\frac{Wab\,(b+2a)}{6\,(a+b)}\end{aligned}\right\}\text{...............}(16).$$

The deflexion in AQ is given by

$$\mathfrak{B}y=\frac{Wabx}{6\,(a+b)}(a+2b)-\tfrac{1}{6}\frac{Wbx^3}{a+b},$$

and that in BQ is given by

$$\mathfrak{B}y = \frac{Wab\,(l-x)}{6\,(a+b)}(b+2a) - \frac{1}{6}\frac{Wa\,(l-x)^3}{a+b}.$$

This can be thrown into the form:—

The deflexion y at P when the weight is at Q and P is in AQ is given by

$$\mathfrak{B}y = \frac{1}{6}\frac{W}{AB}BQ\,.\,AP\,(AB^2 - BQ^2 - AP^2) \;\ldots\ldots\; (17).$$

When P is in BQ the formula becomes

$$\mathfrak{B}y = \frac{1}{6}\frac{W}{AB}AQ\,.\,BP\,(AB^2 - AQ^2 - BP^2) \ldots\ldots\ldots (18).$$

We see that the deflexion at P when the weight is at Q is the same as the deflexion at Q when the weight is at P.

220. Isolated load, built-in ends.

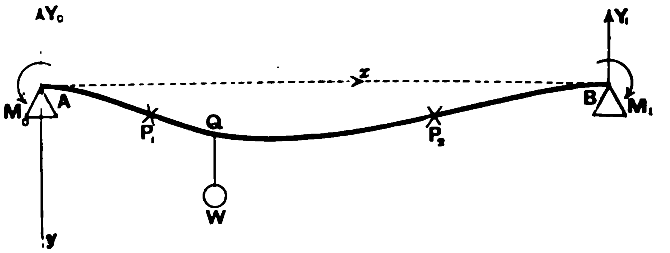


Fig. 25.

When the beam is built in horizontally at both ends let M_0 be the bending moment at A and M_1 at B. Then we have in AQ

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_0 + Y_0x = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (19),$$

and in BQ

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_1 + Y_1(l-x) = 0 \;\ldots\ldots\ldots\ldots\ldots\; (20).$$

Hence in AQ and in BQ

$$\left.\begin{aligned} \mathfrak{B}y &= \tfrac{1}{2}M_0x^2 - \tfrac{1}{6}Y_0x^3, \\ \mathfrak{B}y &= \tfrac{1}{2}M_1(l-x)^2 - \tfrac{1}{6}Y_1(l-x)^3 \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (21),$$

with the conditions

$$\left.\begin{aligned} \tfrac{1}{2}M_0a^2 - \tfrac{1}{6}Y_0a^3 &= \tfrac{1}{2}M_1b^2 - \tfrac{1}{6}Y_1b^3, \\ M_0a - \tfrac{1}{2}Y_0a^2 &= -M_1b + \tfrac{1}{2}Y_1b^2. \end{aligned}\right\}$$

Y_0 and Y_1 are given by taking moments for AB about B and A, and we find

$$\left.\begin{aligned} Y_0(a+b) - Wb + M_1 - M_0 = 0, \\ Y_1(a+b) - Wa + M_0 - M_1 = 0 \end{aligned}\right\} \quad\text{............(22).}$$

Hence we obtain

$$\tfrac{1}{2}M_0a^2(a+b) - \tfrac{1}{6}a^3(Wb + M_0 - M_1) = \tfrac{1}{2}M_1b^2(a+b) - \tfrac{1}{6}b^3(Wa - M_0 + M_1),$$

$$M_0a(a+b) - \tfrac{1}{2}a^2(Wb + M_0 - M_1) = -M_1b(a+b) + \tfrac{1}{2}b^2(Wa - M_0 + M_1);$$

giving $$M_0 = \frac{Wab^2}{(a+b)^2}, \quad M_1 = \frac{Wba^2}{(a+b)^2},$$

and thus $$Y_0 = W\frac{b^2(3a+b)}{(a+b)^3}, \quad Y_1 = W\frac{a^2(3b+a)}{(a+b)^3}.$$

Hence in AQ the deflexion y at a point P is given by

$$\mathfrak{B}y = \tfrac{1}{6}\frac{W}{AB^3}BQ^2 . AP^2(3AQ . BP - BQ . AP) \quad\text{......(23),}$$

and in BQ the deflexion y at a point P is given by

$$\mathfrak{B}y = \tfrac{1}{6}\frac{W}{AB^3}BP^2 . AQ^2(3BQ . AP - AQ . BP) \quad\text{......(24),}$$

and we notice that the deflexion at P when the weight is at Q is the same as the deflexion at Q when the weight is at P.

The points of inflexion are given by $d^2y/dx^2 = 0$, and we find that there is an inflexion at P_1 in AQ where

$$AP_1 = AQ . AB/(3AQ + BQ).$$

In like manner there is an inflexion at P_2 in BQ where

$$BP_2 = BQ . AB/(3BQ + AQ).$$

The point where the central-line is horizontal is given by $dy/dx = 0$. If such a point be in AQ it must be at a distance from A equal to twice AP_1, and for this to happen AQ must be $> BQ$. Conversely if $AQ < BQ$ the point is in BQ at a distance from B equal to twice BP_2.

The verification of these statements will serve as an exercise for the student.

221. The Theorem of Three Moments.

Passing now to the case of a uniform heavy beam resting on any number of supports at the same level we proceed to investigate Clapeyron's theorem of the three moments.

Let...A, B, C...be the supports in order from left to right, let l_{AB} be the length of the span AB,...M_A, M_B... the bending moments at the supports, ... Y_A, Y_B ... the pressures on the supports, A_0, A_1 the shearing forces estimated upwards to left and right of A and indefinitely near to it, B_0, B_1 similar quantities for B and so on. Then $Y_A = A_0 + A_1$, $Y_B = B_0 + B_1$ and so on.

Let w be the load per unit length of the beam.

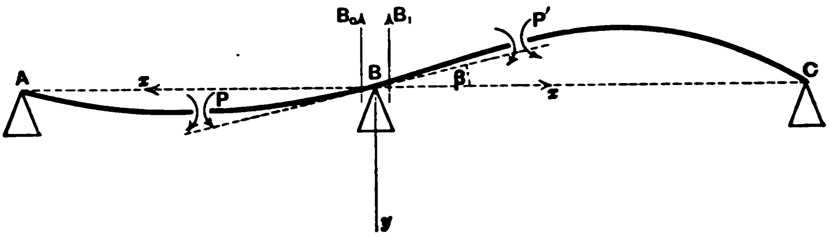


Fig. 26.

In BA measure y downwards and x to the left from B,
in BC measure y downwards and x to the right from B.
Considering the equilibrium of any part BP of BA, and taking moments about P, we have

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_B + B_0x - \tfrac{1}{2}wx^2 = 0 \text{...............(25)}.$$

Integrate this twice, and put $y = 0$ and $dy/dx = \tan\beta$ when $x = 0$, where β is the angle the tangent at B makes with the axis of x, we get

$$\mathfrak{B}(y - x\tan\beta) = \tfrac{1}{2}M_Bx^2 - \tfrac{1}{6}B_0x^3 + \tfrac{1}{24}wx^4 \text{.........(26)}.$$

Again, by taking moments about P' for the part BP' of BC, we have

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_B + B_1x - \tfrac{1}{2}wx^2 = 0 \text{...............(27)}.$$

Integrate this twice and put $y = 0$ and $dy/dx = -\tan\beta$ when $x = 0$, and we get

$$\mathfrak{B}(y + x\tan\beta) = \tfrac{1}{2}M_Bx^2 - \tfrac{1}{6}B_1x^3 + \tfrac{1}{24}wx^4 \text{.........(28)}.$$

Now in (25) $\mathfrak{B}\dfrac{d^2y}{dx^2} = M_A$ when $x = l_{AB}$

and in (27) $\mathfrak{B}\dfrac{d^2y}{dx^2} = M_C$ when $x = l_{BC}$.

Hence
$$\left.\begin{aligned} M_A - M_B + B_0l_{AB} - \tfrac{1}{2}wl_{AB}^2 = 0,\\ M_C - M_B + B_1l_{BC} - \tfrac{1}{2}wl_{BC}^2 = 0\end{aligned}\right\} \text{...............(29)}.$$

These equations give B_0 and B_1 in terms of the bending moments at A, B, C, and the pressure Y_B on the support B is given by the equation

$$Y_B = B_0 + B_1 = \tfrac{1}{2}(l_{AB} + l_{BC})\,w + \frac{M_B - M_A}{l_{AB}} + \frac{M_B - M_C}{l_{BC}} \quad \ldots\text{(30)}.$$

Again in (26) $y = 0$ when $x = l_{AB}$, and in (28) $y = 0$ when $x = l_{BC}$.

Hence
$$\left.\begin{aligned} -\mathfrak{B}\tan\beta &= \tfrac{1}{2}M_B l_{AB} - \tfrac{1}{6}B_0 l_{AB}^2 + \tfrac{1}{24}\,w l_{AB}^3,\\ \mathfrak{B}\tan\beta &= \tfrac{1}{2}M_B l_{BC} - \tfrac{1}{6}B_1 l_{BC}^2 + \tfrac{1}{24}\,w l_{BC}^3 \end{aligned}\right\} \quad \ldots\ldots\text{(31)}.$$

Adding these equations, and substituting for B_0 and B_1 from (29), we find

$$l_{AB}(M_A + 2M_B) + l_{BC}(M_C + 2M_B) = \tfrac{1}{4}w\,(l_{AB}^3 + l_{BC}^3) \quad \ldots\text{(32)},$$

so that the bending moments at three consecutive supports are connected by a relation of invariable form. This is the *Theorem of Three Moments.*

We may express the result by saying that the bending moments at the supports satisfy a linear difference equation of the second order.

The solution of the equation would involve two constants to be determined from the terminal conditions. The bending moment at every support can therefore be calculated. The shearing forces across the sections at the supports can be found from such equations as (29), the pressures on the supports from such equations as (30), the inclination of the central-line of the beam at the supports from such equations as (31), and the deflexion at any point from such equations as (26).

222. Forms of the Equation of Three Moments.

We shall now consider the form assumed by the theorem of three moments for some other distributions of load.

1°. For uniform flexural rigidity when the load on each span is uniformly distributed but the load per unit length varies from span to span, it is easy to shew in the manner of the last article that the equation of three moments becomes

$$l_{AB}(M_A + 2M_B) + l_{BC}(M_C + 2M_B) = \tfrac{1}{4}(w_{AB} l_{AB}^3 + w_{BC} l_{BC}^3) \quad \ldots\text{(33)},$$

and the pressure on the support B is given by

$$Y_B = \tfrac{1}{2}l_{AB}w_{AB} + \tfrac{1}{2}l_{BC}w_{BC} + \frac{M_B - M_A}{l_{AB}} + \frac{M_B - M_C}{l_{BC}} \quad \ldots\text{(34)},$$

where w_{AB} is the load per unit length of the span AB.

2°. **For uniform flexural rigidity and an isolated load W at a point Q between B and C.**

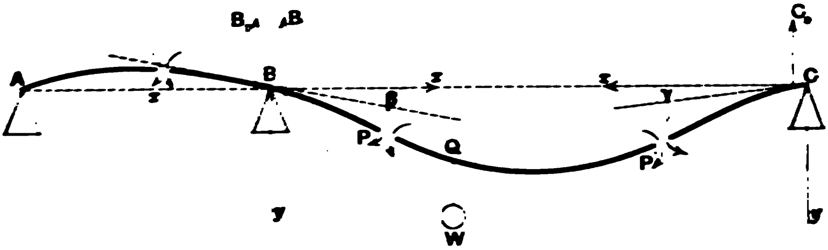


Fig. 27.

Let $BQ = b$ and $CQ = c$.

In BQ measure y downwards and x to the right;

in CQ measure y downwards and x to the left.

Suppose the tangents at B and C to make angles β and γ with the horizontal.

The equation of moments for any part BP of BQ is

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_B + B_1 x = 0 \qquad \text{(35)},$$

and the equation of moments for any part CP' of CQ is

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_C + C_0 x = 0 \qquad \text{(36)};$$

from which we find

$$\left.\begin{array}{ll} \text{in } BQ & \mathfrak{B}(y - x\tan\beta) = \frac{1}{2}M_B x^2 - \frac{1}{6}B_1 x^3, \\ \text{in } CQ & \mathfrak{B}(y - x\tan\gamma) = \frac{1}{2}M_C x^2 - \frac{1}{6}C_0 x^3 \end{array}\right\} \qquad \text{(37)}.$$

When $x = b$ in BQ, and $x = c$ in CQ, the y's are the same and the sum of the dy/dx's is zero. Hence

$$\left.\begin{array}{l} \mathfrak{B}(b\tan\beta - c\tan\gamma) = -\frac{1}{2}M_B b^2 + \frac{1}{2}M_C c^2 + \frac{1}{6}B_1 b^3 - \frac{1}{6}C_0 c^3, \\ \mathfrak{B}(\tan\beta + \tan\gamma) = -(M_B b + M_C c) + \frac{1}{2}(B_1 b^2 + C_0 c^2) \end{array}\right\} \qquad \text{(38)}.$$

Now, by taking moments about C and B for BC, we find

$$\left.\begin{array}{l} B_1(b+c) - Wc + M_C - M_B = 0, \\ C_0(b+c) - Wb + M_B - M_C = 0 \end{array}\right\} \qquad \text{(39)}.$$

Thus

$$\left.\begin{array}{l} \mathfrak{B}(b+c)\tan\beta = \frac{1}{6}Wbc(b+2c) - \frac{1}{6}(2M_B + M_C)(b+c)^2, \\ \mathfrak{B}(b+c)\tan\gamma = \frac{1}{6}Wbc(c+2b) - \frac{1}{6}(2M_C + M_B)(b+c)^2 \end{array}\right\} \qquad \text{(40)}.$$

Now in the span AB of length a preceding B measuring x from B we have

$$\mathfrak{B}\frac{d^2y}{dx^2}-M_B+B_0x=0\text{.....................}(41),$$

and

$$M_A-M_B+B_0a=0\text{.....................}(42);$$

integrating (41), and putting $y=0$ and $dy/dx=-\tan\beta$ when $x=0$, we find

$$\mathfrak{B}(y+x\tan\beta)=\tfrac{1}{2}M_Bx^2-\tfrac{1}{6}(M_B-M_A)x^3/a\text{......}(43),$$

and since $y=0$ when $x=a$,

$$\mathfrak{B}\tan\beta=\tfrac{1}{6}a(2M_B+M_A)\text{..................}(44).$$

In like manner by considering the span of length d succeeding C we have

$$\mathfrak{B}\tan\gamma=\tfrac{1}{6}d(2M_C+M_D)\text{..................}(45).$$

By (40) and (44) we find the equation of three moments for A, B, C in the form

$$AB(M_A+2M_B)+BC(M_C+2M_B)=\frac{W}{BC}BQ\,.\,CQ(BQ+2CQ)\text{...}(46),$$

and the equation for B, C, D is in like manner

$$BC(M_B+2M_C)+CD(M_D+2M_C)=\frac{W}{BC}BQ\,.\,CQ(CQ+2BQ)\text{...}(47),$$

the weight W being at Q between B and C.

223. Equation of Three Moments generalised.

The theorem of three moments may be generalised so as to include the cases of variable flexural rigidity and variable load[1]. For this we begin with the case where there is no load between A and B or between B and C, but the flexural rigidity $\mathfrak{B}$ is a function of x.

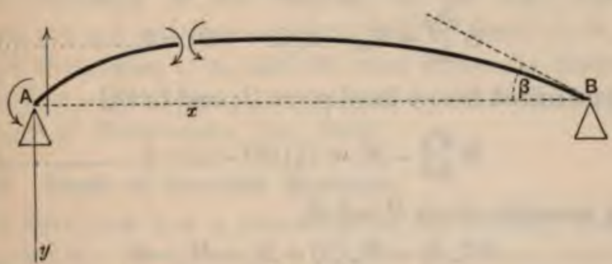


Fig. 28.

[1] Webb, *Camb. Phil. Soc. Proc.* VI. 1886.

In the span AB we have, measuring x towards B,

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_A + \frac{M_A - M_B}{AB}x = 0 \dots\dots\dots\dots\dots(48),$$

since as in equation (42) of the last article we may express the shearing force to the right of A by taking moments about B.

Now multiplying this equation by $x/\mathfrak{B}$ and integrating, we find

$$x\frac{dy}{dx} - y = M_A\int_0^x \frac{x(AB - x)}{AB}\frac{dx}{\mathfrak{B}} + M_B\int_0^x \frac{x^2}{AB}\frac{dx}{\mathfrak{B}} \dots(49).$$

Since $y = 0$ and $dy/dx = \tan\beta$ when $x = AB$, we have

$$\tan\beta = M_A\int_0^{AB} \frac{x(AB - x)}{AB^2}\frac{dx}{\mathfrak{B}} + M_B\int_0^{AB} \frac{x^2}{AB^2}\frac{dx}{\mathfrak{B}} \dots(50).$$

In the next span BC take C as origin and measure x towards B and we find

$$-\tan\beta = M_C\int_0^{BC} \frac{x(BC - x)}{BC^2}\frac{dx}{\mathfrak{B}} + M_B\int_0^{BC} \frac{x^2}{BC^2}\frac{dx}{\mathfrak{B}} \dots(51).$$

Now measuring x towards A from a fixed point O in BA produced we have

$$M_A\int_{OA}^{OB} \frac{(x - OA)(OB - x)}{AB^2}\frac{dx}{\mathfrak{B}} + M_C\int_{OB}^{OC} \frac{(x - OB)(OC - x)}{BC^2}\frac{dx}{\mathfrak{B}}$$
$$+ M_B\int_{OA}^{OB} \frac{(x - OA)^2}{AB^2}\frac{dx}{\mathfrak{B}} + M_B\int_{OB}^{OC} \frac{(OC - x)^2}{BC^2}\frac{dx}{\mathfrak{B}} = 0 \dots(52).$$

This is the equation of three moments for this case and we shall denote the left-hand member by $[ABC]$.

If now we suppose that there is a weight W at a point Q between B and C and no load between A and B, we have in BQ

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_B + B_1(x - OB) = 0 \dots\dots\dots\dots\dots(53),$$

x being measured from a fixed point O; and in CQ

$$\mathfrak{B}\frac{d^2y}{dx^2} - M_C + C_0(OC - x) = 0 \dots\dots\dots\dots\dots(54).$$

Also, by moments about C and B,

$$\left.\begin{aligned} BC.B_1 - W.CQ + M_C - M_B &= 0, \\ BC.C_0 - W.BQ + M_B - M_C &= 0 \end{aligned}\right\} \dots\dots\dots\dots(55).$$

We multiply equation (53) by $(x - OB)/\mathfrak{B}$ and integrate from

OB to OQ, and equation (54) by $(OC-x)/\mathfrak{B}$ and integrate from OQ to OC, and we get

$$\left.\begin{aligned} BQ\left(\frac{dy}{dx}\right)_Q - y_Q &= M_B\int_{OB}^{OQ}(x-OB)\frac{dx}{\mathfrak{B}} - B_1\int_{OB}^{OQ}(x-OB)^2\frac{dx}{\mathfrak{B}},\\ -CQ\left(\frac{dy}{dx}\right)_Q - y_Q &= M_C\int_{OQ}^{OC}(OC-x)\frac{dx}{\mathfrak{B}} - C_0\int_{OQ}^{OC}(x-OC)^2\frac{dx}{\mathfrak{B}} \end{aligned}\right\} \quad\text{.................}(56)$$

By subtracting these we find an expression for $(dy/dx)_Q$. But another expression for this can be found by dividing (53) by $\mathfrak{B}$ and integrating; we get thus

$$\left(\frac{dy}{dx}\right)_Q - \tan\beta = M_B\int_{OB}^{OQ}\frac{dx}{\mathfrak{B}} - B_1\int_{OB}^{OQ}(x-OB)\frac{dx}{\mathfrak{B}}\text{...}(57).$$

Equating the two expressions for $(dy/dx)_Q$ we obtain the equation

$$\tan\beta = -M_B\int_{OB}^{OQ}\frac{OC-x}{BC}\frac{dx}{\mathfrak{B}} - M_C\int_{OQ}^{OC}\frac{OC-x}{BC}\frac{dx}{\mathfrak{B}}$$
$$+B_1\int_{OB}^{OQ}\frac{(OC-x)(x-OB)}{BC}\frac{dx}{\mathfrak{B}} + C_0\int_{OQ}^{OC}\frac{(OC-x)^2}{BC}\frac{dx}{\mathfrak{B}}\text{...}(58).$$

But by equation (50) we have another expression for $\tan\beta$, and, using (55) and equating these, we obtain an equation which can be written

$$[ABC] = W.QC\int_{OB}^{OQ}\frac{(OC-x)(x-OB)}{BC^2}\frac{dx}{\mathfrak{B}} + W.QB\int_{OQ}^{OC}\frac{(OC-x)^2}{BC^2}\frac{dx}{\mathfrak{B}} \quad\text{.................}(59).$$

where the left-hand member is the quantity expressed on the left-hand side of equation (52).

If there be any number of loads the results may be found by summation.

The extension of the theorem to the case where the supports are not on the same level will be found in M. Lévy's *Statique Graphique*, t. II., and the case where the supports are slightly compressible has been treated by Prof. Pearson in the *Messenger of Mathematics*, XIX. 1890.

224. Basis of Graphic Method[1].

We have seen how a knowledge of the bending moment at every point of a beam resting on supports leads to a determination of the form assumed by the beam and the pressures on the supports,

[1] For references see Introduction p. 12.

and further how the bending moment at any point can be determined from those at the extremities of the span containing it, and we have given the equations by which the bending moments at consecutive supports are connected. We shall now shew how the calculation can be superseded by a graphical construction.

The construction in question rests upon two theorems as follows:—

Theorem I. The form assumed by a beam in which the bending moment is given is identical with that of a catenary or funicular curve when the load per unit of length of its horizontal projection is proportional to the given bending moment.

For the equations of equilibrium of the string under a load Gdx on an element whose projection on the axis of x is dx are

$$\left.\begin{aligned} T\frac{dx}{ds} &= \text{const.} = \tau \text{ say} \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) &= G\frac{dx}{ds} \end{aligned}\right\}$$

where T is the tension and ds an element of the arc. The elimination of T between these equations leads to the equation

$$\tau\frac{d^2y}{dx^2} = G$$

of the same form as the equation of the beam

$$\mathfrak{B}\frac{d^2y}{dx^2} = G.$$

Theorem II. The tangents at the extremities of any span are the same as the tangents to the funicular that would be obtained by replacing the forces Gdx on that span by any equivalent system.

For if we measure x from one extremity of a span of length l we have by integration of the above equation

$$x\frac{dy}{dx} - y = \int_0^x x\frac{G}{\mathfrak{B}}dx$$

and thus

$$l\left.\overline{\frac{dy}{dx}}\right|_l = \int_0^l x\frac{G}{\mathfrak{B}}dx$$

and this integral is the same for the actual system of forces proportional to Gdx and for any equivalent distribution of force.

225. The equivalent system of forces.

Let AB be any span supposed subject to a given distribution of load, and draw a curve whose ordinate at any point represents the bending moment at the corresponding point of AB. This may be conveniently effected as follows:—Draw from A and B vertical

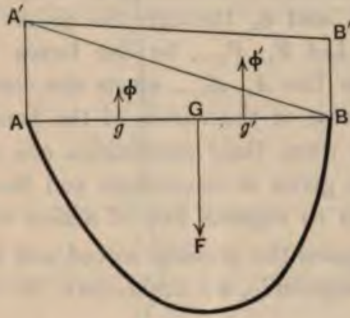


Fig. 29.

lines AA' and BB' proportional to the bending moments at A and B, and draw through A, B the curve whose ordinates are the bending moments at the points of AB when this span is isolated and simply supported at A and B and is under the given distribution of load. Let the verticals through the centroids of the triangles $AA'B$ and $BA'B'$ meet AB in g and g', they are the vertical trisectors of the line AB, and let the vertical through the centroid of the curve of the bending moment when A and B are simply supported meet AB in G.

The funicular whose extreme sides are the directions of the central-line of the beam through A and B is that for vertical forces ϕ, ϕ', and F acting through g, g', and G proportional respectively to the areas of $AA'B$, $BA'B'$, and to the area of the curve of the bending moment when A and B are simply supported. This funicular being constrained to pass through A and B is completely determinate when all the bending moments are given.

If funiculars be constructed in this way for each span then the extreme sides of two consecutive funiculars which meet in a point of support, as B, must be in the same straight line. This remark leads to the determination of the bending moments at the supports.

226. Development of the method.

Suppose to fix ideas that $A_0, A_1, A_2 \ldots$ are the points of support, A_0 being simply supported, and that for any span $A_r A_{r+1}$ the curve of the bending moment when A_r and A_{r+1} are simply supported has been drawn. Let $\phi'_1, \phi_2, \phi'_2 \ldots$ be the forces that act in the vertical trisectors of $A_0 A_1, A_1 A_2 \ldots, \phi_1'$ acting through the point g_1' nearer to A_1, ϕ_2, and ϕ_2' through the points g_2, and g_2' and so on (see fig. 30). Let $F_1, F_2 \ldots$ be the forces that act through $G_1, G_2 \ldots$ where the line $A_0, A_1 \ldots$ meets the verticals through the centroids of the areas of the curves of the bending moments for the various spans when their extremities are simply supported. Then $F_1, F_2 \ldots$ are given in magnitude and line of action, while $\phi_1', \phi_2 \ldots$ are given as regards line of action only.

If now we supposed the problem solved, and these forces consequently fixed in magnitude, we could draw the funiculars for the various spans.

Suppose 1, 2, 3... are the sides of the funiculars of which 1, 2, and 3 belong to the span $A_0 A_1$, 3, 4, 5 and 6 to the span $A_1 A_2$ and so on. The sides 1, 3, 6, 9... pass through the points of support.

Let the sides 2 and 4 meet in V_1. Then in the triangle whose sides are 2, 3, and 4 the vertex V_1 must be on the line of action of the resultant of ϕ'_1 and ϕ_2, *i.e.* on a fixed vertical meeting A_1 in α_1 where $\alpha_1 g_2 = A_1 g'_1$. The triangle whose sides are 2, 3, and 4 therefore has its vertices on three fixed lines. We can shew that its sides pass through three fixed points. For the side 3 passes through the fixed point A_1. If the side 2 meet the vertical through A_0 in C_2 we shall see that C_2 is a fixed point. For the triangle whose sides are 1, 2, and the vertical through A_0 is a triangle of forces for the point of intersection of 1 and 2, and $A_0 C_2$ represents F_1 on the scale on which we represent forces by lines. Since two sides 2 and 3 of the triangle formed by 2, 3, 4 pass through fixed points C_2 and A_1 while the vertices move on three fixed parallel lines, the remaining side 4 also passes through a fixed point C_4 which is on the line $C_2 A_1$. This point C_4 may be found by trial by drawing any triangle which fulfils the other five conditions.

In the same way we may prove that the triangle whose sides are 5, 6, 7 has its vertices in three fixed vertical lines (viz. those

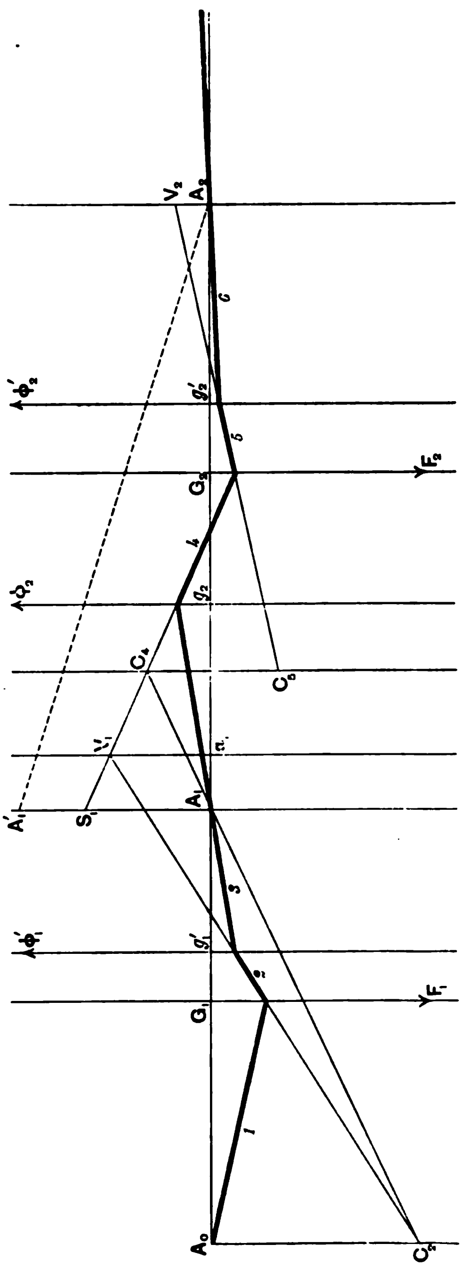


Fig. 30.

through g'_2, g_3, and α_2, where $\alpha_2 g_3 = A_2 g'_2$,) and that its sides pass through three fixed points. In fact to find the fixed point C_5 on the side 5 we have to take on the vertical through C_4 a length $C_4 C_5$ which has to the line representing the force F_2 the same ratio as the horizontal distance of G_2 from the vertical through C_4 has to the horizontal distance $A_0 G_1$ of G_1 from the vertical through A_0. For the triangle formed by 4, 5, and the vertical through C_4 is then a triangle of forces for the point of intersection of 4 and 5, but the scale on which its sides represent forces is changed in comparison with that of the triangle formed by 1, 2 and the vertical through A_0 in the ratio stated. C_5 being determined the fixed point C_7 on the side 7 may be found by trial, and we can in this way determine two series of points $C_2, C_5, \ldots C_{3n-1}$ and $C_4, C_7, \ldots C_{3n-2}$.

Consider the case where the further extremity A_n of the nth span is freely supported. Find the above two series of points. Join C_{3n-1} to A_n, this determines the last side of the funicular. From the point where it meets the line of action of F_n draw a line to C_{3n-2}, and produce this to meet the vertical through α_{n-1}[1] in V_{n-1}. Join V_{n-1} to C_{3n-4} and proceed in the same way, and we obtain the funicular. The side $(3n-3)$ of the funicular will be found by joining the points where $(3n-2)$ meets the vertical through g_n and where $(3n-4)$ meets the vertical through g'_{n-1}. This side must pass through A_{n-1}. The sides $(3n-6)$, $(3n-9)\ldots$ may be constructed in like manner.

The bending moments at the supports may also be found graphically. Let the vertical through A_r meet the side $(3r+1)$ in S_r, then $A_r S_r / A_r A^2_{r+1}$ is proportional to the bending moment at A_r. We have seen that if $A_r A'_r$ be drawn to represent the bending moment at A_r then ϕ_r is proportional to the area of the triangle $A_r A'_r A_{r+1}$ and acts vertically through its centroid. Thus if M_r be the bending moment at A_r

$$\phi_r \propto M_r \,.\, A_r A_{r+1}.$$

But since the sides $3r$, $(3r+1)$, and the vertical through A_r are a triangle of forces for the intersection of 3 and 4 and the breadth of this triangle is $\frac{1}{3} A_r A_{r+1}$ we have

$$\phi_r \propto A_r S_r / A_r A_{r+1}.$$

Hence $$M_r \propto A_r S_r / A_r A^2_{r+1}.$$

[1] α_{n-1} is such that $\alpha_{n-1} g_n = A_{n-1} g'_{n-1}$, and $\alpha_{n-1} g'_{n-1} = A_{n-1} g_n$.

For details in regard to the scale on which the lines are drawn to represent forces, and the extensions of the method to cases where the extremities are built in, or the supports are not all on the same level, the reader is referred to M. Lévy's *Statique Graphique*, t. II. A paper by Professors Perry and Ayrton in *Proc. R. S. Lond.* Nov. 1879 may also be consulted.

227. The Elastica.

We shall now consider the problem presented by a thin rod which is held in the shape of a plane curve by forces and couples applied at its ends alone. The forms of the elastic central-line under this condition are the curves of a certain family to which the name of *Elastica* has been given.

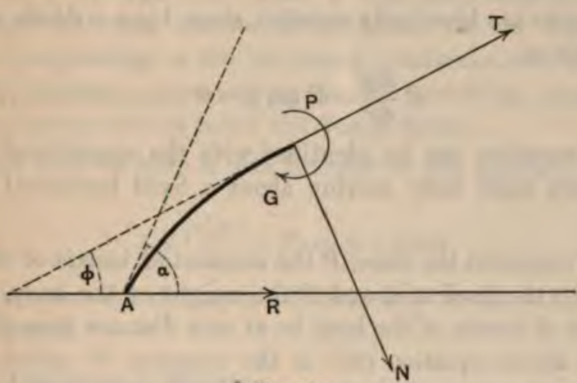


Fig. 31.

As in art. 214 let T, N, G be the tension, the shearing force, and the flexural couple at any point P of the strained elastic central-line, and let ϕ be the angle which the tangent at P makes with a fixed line; as the figure is drawn the curvature of the elastic central-line at P is $-d\phi/ds$, and the couple G is $-B\,d\phi/ds$ where B is the flexural rigidity.

Let A be any fixed point of the rod, and R the resultant of N and T at A, if we consider the equilibrium of the part of the rod between A and P, and resolve all the forces acting on this part in the direction of R, we shall find that N and T at P must have a resultant which is equal and opposite to R. Hence the magnitude and direction of R are fixed, and they are identical with the magnitude and direction of the force applied to the end of the rod.

If therefore ϕ be the angle which the tangent to the rod at P makes with the line of action of the force applied at the end of the rod we have

$$T = -R\cos\phi,\quad N = -R\sin\phi \quad \text{............}(60).$$

The general equations of equilibrium are known from art. 214 to be

$$\frac{dT}{ds} - \frac{N}{\rho} = 0,$$

$$\frac{dN}{ds} + \frac{T}{\rho} = 0,$$

$$\frac{dG}{ds} + N = 0.$$

The first two are identically satisfied since $1/\rho = -d\phi/ds$, and the third gives us

$$B\frac{d^2\phi}{ds^2} + R\sin\phi = 0 \quad \text{....................}(61).$$

This equation can be identified with the equation of motion of a heavy rigid body moving about a fixed horizontal axis as follows:

Let s represent the time, B the moment of inertia of the rigid body about the fixed axis, and R the weight of the body, and let the centre of inertia of the body be at unit distance from the fixed axis, the above equation (61) is the equation of motion of the body when the plane through the centre of inertia and the fixed axis makes an angle ϕ with the vertical.

Hence if a rigid pendulum of weight R be constructed to have its moment of inertia about a certain axis equal to B, and its centre of inertia at unit distance from that axis, and swing about that axis under gravity, and, if the centre of suspension of the pendulum move along the tangent to the elastic central-line of the bent rod with unit velocity, a line fixed in the pendulum will be always a tangent to the elastic central-line at the point of suspension of the pendulum.

This is a case of a general theorem known as Kirchhoff's *Kinetic Analogue*. (See below art. 236.)

The motion of the rigid pendulum is the same as that of a simple circular pendulum of length gB/R, where g is the acceleration due to gravity. As is well known, the integration of the

equation of motion of the pendulum involves elliptic functions, and there are two cases according as the pendulum oscillates or makes complete revolutions. The corresponding forms of the elastica are distinguished by the circumstance that in the first case there are points of inflexion, and in the second case there are none, since the angular velocity of the pendulum is identical with the curvature of the elastica. We shall be able to give a particular form to our results by assuming that in all cases the rod is held in the shape of the elastica by equal and opposite forces R applied to its two extremities. In the first case the forces will be supposed directly applied, and the ends of the rod will be inflexions on the elastica; in the second case they will be supposed to be applied by means of rigid arms attached to the ends. The two cases will be distinguished as the *inflexional elastica* corresponding to the oscillating pendulum, and the *non-inflexional elastica* corresponding to the revolving pendulum. The line of action of R is called the *line of thrust.*

In both cases we have a first integral of the equation (61) in the form

$$\tfrac{1}{2} B \left(\frac{d\phi}{ds}\right)^2 = R \cos\phi + \text{const.} \quad \text{............(62)},$$

and the two cases are distinguished according to the value taken by the constant.

228. Inflexional Elastica.

Suppose first that the pendulum of the kinetic analogue oscillates through an angle 2α. We have

$$\tfrac{1}{2} B \left(\frac{d\phi}{ds}\right)^2 = R(\cos\phi - \cos\alpha) \quad \text{...............(63)}.$$

Introducing elliptic functions of an argument u and modulus k, where

$$u = s\sqrt{(R/B)}, \quad k = \sin\tfrac{1}{2}\alpha \quad \text{..................(64)},$$

we find

$$\left.\begin{aligned} \frac{d\phi}{ds} &= 2k\sqrt{\frac{R}{B}}\,\mathrm{cn}\,(u+K), \\ \sin\tfrac{1}{2}\phi &= k\,\mathrm{sn}\,(u+K) \end{aligned}\right\} \quad \text{...............(65)}.$$

Now referring to fixed axes of x and y, of which the axis x coincides with the line of action of the force R we have

$$\frac{dx}{ds} = \cos\phi, \qquad \frac{dy}{ds} = \sin\phi.$$

Of these the first gives us

$$x = \int [1 - 2k^2 \operatorname{sn}^2 (u + K)]\, ds$$

$$= -\sqrt{\frac{B}{R}}\, u + 2\sqrt{\frac{B}{R}} \{E \operatorname{am} (u + K) - E \operatorname{am} K\} \quad\ldots\ldots(66),$$

where E is the elliptic integral of the second kind defined by

$$E \operatorname{am} u = \int_0 \operatorname{dn}^2 u \, du,$$

the path of integration being real.

The second equation gives us

$$y = \quad 2k \int \operatorname{sn} (u + K) \operatorname{dn} (u + K)\, ds,$$

or

$$y = -2k \sqrt{\frac{B}{R}} \operatorname{cn} (u + K)^1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(67).$$

The constants have been so chosen that the origin is at a point of inflexion on the curve. Since the flexural couple at the ends vanishes when the rod is subjected to terminal forces only, the origin may be taken at an end of the rod in this case. It follows that with our choice of axes the axis of x is the line of thrust.

Wherever the curve cuts the line of thrust we have a point of inflexion, and the tangent is inclined to the line of action of R at an angle α.

The form of the curve depends on the value of α, and we shall suppose α to vary from 0 to π and find the corresponding forms of the curve[2]. When there are two inflexions the force R, the length l, and the angle α are connected by the relation

$$l\sqrt{(R/B)} = 2K\,(\text{mod.} \sin \tfrac{1}{2}\alpha).$$

Forms with more than two inflexions are probably unstable.

[1] Notice by way of verification that $Ry = -Bd\phi/ds = G$ as is obviously the case.

[2] Hess, 'Ueber die Biegung und Drillung eines unendlich dünnen elastischen Stabes mit zwei gleichen Widerständen...'. *Math. Ann.* xxv. 1885. The classification was first made by Euler. (See Introduction.)

1°. When $\alpha = 0$, ϕ is constantly zero and the rod remains straight.

2°. When $\alpha < \frac{1}{2}\pi$, the curve always makes an acute angle with the line of action of R, and we have the figure

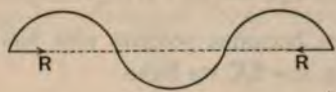


Fig. 32.

This includes the case where the rod is infinitely little bent, corresponding to the case where the pendulum makes indefinitely small oscillations. In this case ϕ is always very small and it can be easily shewn that the equation of the curve is of the form

$$y \propto \sin\{x\sqrt{(R/B)}\},$$

so that the limiting form, when α is indefinitely diminished, is a curve of sines of very small amplitude.

3°. When $\alpha = \frac{1}{2}\pi$, the curve corresponds to the limiting case of the above when the inflexional tangents are at right angles to the line of action of R.

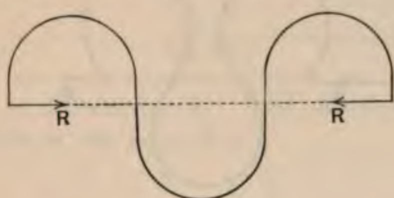


Fig. 33.

4°. When $\alpha > \frac{1}{2}\pi$, we may take $\pi - \alpha = \beta$ and then

$$\cos\phi - \cos\alpha = \cos\phi + \cos\beta.$$

We thus obtain

$$\cos\phi + \cos\beta = 2k^2 \operatorname{cn}^2(u + K),$$

and as u increases $\cos\phi$ is at first negative and approaching zero. Also $\cos\phi$ vanishes when

$$\operatorname{cn}^2(u + K) = \cos\beta/(1 + \cos\beta),$$

or

$$\operatorname{dn}^2(u + K) = \tfrac{1}{2}.$$

4—2

Suppose u_1 is the least positive value of u given by this equation. Then the x of the curve vanishes with u, and as u increases x is at first negative and its absolute value is a maximum when $u = u_1$. When

$$u = 2\{E\,\mathrm{am}\,(u + K) - E\,\mathrm{am}\,K\},$$

x vanishes. It then becomes positive and has a maximum when $u = 2K - u_1$. When $u = 2K$ we have

$$x_{2K} = 2x_K = 2\sqrt{\frac{B}{R}}\,(2E\,\mathrm{am}\,K - K)\text{............(68)}.$$

Further we have $\qquad x_{2K \pm u} = x_{2K} \pm x_u.$

There are three cases according as x_K is positive, negative or zero.

(1) $x_K > 0$ or $2E\,\mathrm{am}\,K > K$. In this case x_K, x_{2K}, ... are all positive and the curve proceeds in the positive direction of the axis of x. There are three subcases.

(α) $x_K > -x_{u_1}$ the curve does not cut itself.

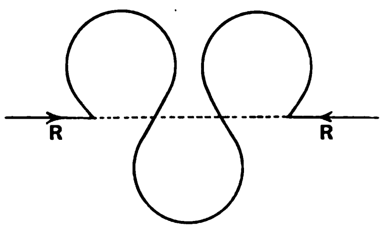


Fig. 34.

(β) $x_K = -x_{u_1}$ the successive parts of the curve touch each other.

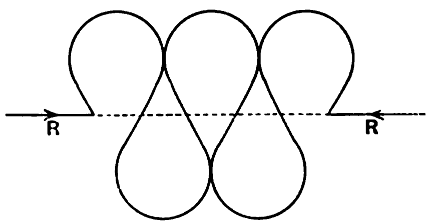


Fig. 35.

(γ) $x_K < -x_{\omega_1}$ the successive parts of the curve cut each other.

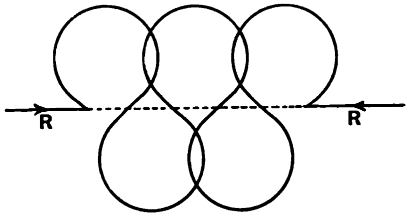


Fig. 36.

(2) $x_K = 0$ or $2E \operatorname{am} K = K$. This is the limiting case of the last when the double points and inflexions all come to coincide with the origin; the curve may consist of several exactly equal and similar parts lying one over another.

The value of α for this is about $129^\circ{\cdot}3$.

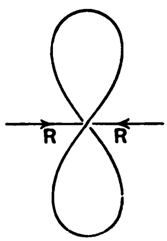


Fig. 37.

(3) $x_K < 0$ or $2E \operatorname{am} K < K$. The curve proceeds in the negative direction of the axis of x.

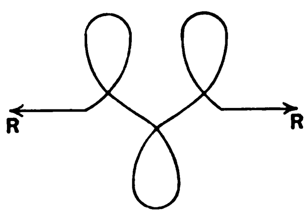


Fig. 38.

5°. When $\alpha = \pi$. This is the limiting case of the last when

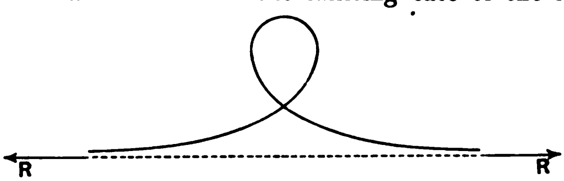


Fig. 39.

the rod of very great (ultimately infinite) length forms a single loop.

229. Non-inflexional Elastica.

Next suppose the pendulum of the kinetic analogue makes complete revolutions. The equation (50) takes the form

$$\tfrac{1}{2}B\left(\frac{d\phi}{ds}\right)^2 = R\cos\phi + R\left(1 + 2\,\frac{1-k^2}{k^2}\right) \quad\text{.........(69)},$$

where k is less than unity.

Then we have, writing u for $\frac{1}{k}s\sqrt{(R/B)}$,

$$\left.\begin{aligned} \frac{d\phi}{ds} &= \frac{2}{k}\sqrt{\frac{R}{B}}\,\mathrm{dn}\,u \ (\mathrm{mod}\,k), \\ \sin\tfrac{1}{2}\phi &= \mathrm{sn}\,u \end{aligned}\right\} \quad\text{...............(70)},$$

and the values of x and y are given by the equations

$$\left.\begin{aligned} x &= k\sqrt{\frac{B}{R}}\left[\left(1-\frac{2}{k^2}\right)u + \frac{2}{k^2}E\,\mathrm{am}\,u\right], \\ y &= -\frac{2}{k}\sqrt{\frac{B}{R}}\,\mathrm{dn}\,u \end{aligned}\right\} \quad\text{.........(71)}.$$

Since there are no inflexions these forms are not possible without terminal couple.

The form of the curve is

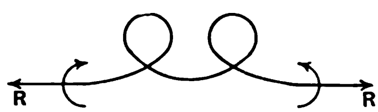


Fig. 40.

The limiting case when $k=1$ is that mentioned in no. 5° of the last article. It corresponds to the case where the bob of the pendulum starts from the lowest position with velocity which will just carry it to the highest position in an infinite time.

We have in this case

$$\tfrac{1}{4}\left(\frac{d\phi}{ds}\right)^2 = \frac{R}{B}\cos^2\frac{\phi}{2} \quad\text{....................(72)},$$

and writing u for $s\sqrt{(R/B)}$ we obtain

$$\left.\begin{aligned} \frac{d\phi}{ds} &= 2\sqrt{\frac{R}{B}}\,\mathrm{sech}\,u, \\ \sin\tfrac{1}{2}\phi &= \tanh u \end{aligned}\right\} \quad\text{.................(73)}.$$

From these
$$\left.\begin{aligned} x &= \sqrt{\frac{B}{R}}(-u + 2\tanh u), \\ y &= -2\sqrt{\frac{B}{R}}\operatorname{sech} u \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(74).$$

The figure is given in no. 5° of the last article.

The particular case where R vanishes and the rod is held bent by terminal couples corresponds to the motion of a pendulum whose centre of suspension coincides with its centre of inertia. The terminal couple corresponds to the angular momentum of the pendulum, and the flexural couple is the same at all points of the rod. Since $d\phi/ds$ is constant the curve is a circle.

The circle and the series of figures given in the last two articles include all the typical forms of plane curves in which a thin rod initially straight can be held by terminal forces and couples only.

It may be shewn that the same series of curves include all the plane forms in which a rod whose elastic central-line is initially circular can be held by terminal forces and couples. The proof is left to the reader.

230. Rod bent by Normal Forces.

We have just seen that if no bodily forces be directly applied to a thin rod the plane forms in which it can be held belong to a certain definite family of curves. In order to hold the rod so that its elastic central-line is of any form which is not one of the elastica family, forces will have to be applied directly to its elements. We can now see that purely normal forces are always adequate to hold the rod in any required form, or in other words that the form can be obtained by bending the rod against a smooth rigid cylinder.

The equations of equilibrium under purely normal forces Y per unit of length are

$$\left.\begin{aligned} &\frac{dT}{ds} - \frac{N}{\rho} = 0, \\ &\frac{dN}{ds} + \frac{T}{\rho} + Y = 0, \\ &\frac{dG}{ds} + N = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75),$$

in which $G = B/\rho$, so that $N = -B\dfrac{d}{ds}\left(\dfrac{1}{\rho}\right)$.

Substituting for N in the first equation and integrating we have

$$T+\tfrac{1}{2}B/\rho^2=T_0 \quad\text{.........................(76)},$$

where T_0 is the tension at a point of inflexion.

The second equation now gives

$$-Y=T_0/\rho-\tfrac{1}{2}B/\rho^3-B\frac{d^2}{ds^2}\left(\frac{1}{\rho}\right) \quad\text{............(77)};$$

so that the required normal force and the tension at any point are determined in terms of given quantities and a single constant T_0 depending on the terminal conditions.

As an example we may find the pressure when a wire is fitted round an elliptic cylinder of semi-axes a, b and its ends joined. The pressure at either end of the minor axis vanishes, and the pressure at either end of the major axis is

$$\tfrac{1}{2}B(a^2-b^2)(5a^4-a^2b^2+5b^4)/a^3b^6.$$

[Math. Tripos, 1885.

231. Uniform Normal Force.

In further illustration of the general equations we may consider the forms in which a rod can be held bent by *uniform* normal pressure. For this case M. Lévy[1] has given an interesting theorem as follows:—

When a plane rod or wire is held in a plane form by uniform normal pressure, the tension and shearing force across the normal section through any point of the strained elastic central-line have a resultant which is perpendicular and proportional to the line joining the point to a fixed point in the plane.

Let Y be the pressure per unit length, T and N the tension and shearing force at any point P of the strained elastic central-line and take as temporary axes of x and y the tangent and normal at P, and let F be the resultant of N and T.

The equation of a line through P perpendicular to the direction of F is

$$\frac{y}{T}=-\frac{x}{N}.$$

The equation of a line through a neighbouring point P' perpendicular to the direction of F there, the new F being the

[1] Liouville's *Journal*, x. 1884.

resultant of $T+dT$ and $N+dN$ along the tangent and normal at P', is

$$\frac{y}{T+dT-Nd\phi}=\frac{x-\rho d\phi}{-N-dN-Td\phi},$$

where ρ is the radius of curvature of the strained elastic central line at P, and $d\phi$ the angle of contingence of the same curve.

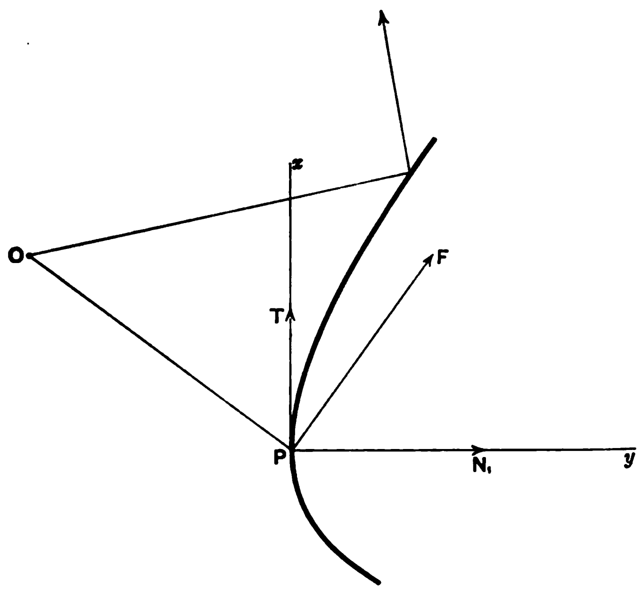


Fig. 41.

Now by the equations of equilibrium the equation of the line through P' is

$$\frac{y}{T}=\frac{x-\rho d\phi}{-N+Y\rho d\phi},$$

and this line meets the corresponding line through P in the point

$$x=N/Y, \qquad y=-T/Y.$$

Hence, if through P a line PO be drawn perpendicular and proportional to F so that $Y.OP=F$, the line similarly drawn through a neighbouring point P' will terminate at the same point O. The point O may be called the *elastic centre of pressure* for the curve.

Now taking the fixed point O as the origin of polar coordinates we have for the shearing force N

$$N = -F\frac{dr}{ds} = -rY\frac{dr}{ds},$$

and by the third of equations (75) we have for the flexural couple G

$$\frac{dG}{ds} = -N = rY\frac{dr}{ds},$$

so that $$G = \tfrac{1}{2}Yr^2 + \text{const.}$$

This equation has the same form whether the rod be initially straight or initially curved, and supposing that initially the rod was of curvature $1/\rho_0$, a given function of s, we have

$$B(1/\rho - 1/\rho_0) = \tfrac{1}{2}Yr^2 + \text{const.}$$

which is an equation to determine the form of the curve.

M. Lévy considers particularly the case of a rod initially circular, for which ρ_0 is constant and equal to a. If p be the perpendicular from the point O on the tangent to the strained elastic central-line ρ is rdr/dp, and we have the equation

$$\frac{dp}{rdr} = \frac{1}{a} + \frac{C}{B} + \tfrac{1}{2}\frac{Yr^2}{B},$$

or $$p = \tfrac{1}{2}r^2\left(\frac{1}{a} + \frac{C}{B}\right) + \tfrac{1}{8}r^4\frac{Y}{B} + C' \quad\text{...............(78)},$$

where C and C' are constants. This is the general (p, r) equation of the curve in which an initially circular rod or wire can be held by uniform normal pressure Y per unit length.

It has been shewn by M. Lévy that the arc s and the vectorial angle θ of the curve are elliptic integrals with argument r^2, and conversely r^2 may be expressed either in terms of elliptic functions of s or of elliptic functions of θ.

If the wire initially form a complete circular ring two conditions must be satisfied, viz. the real period of r^2 as a function of θ must be $2\pi/n$ where n is an integer, and the arc s corresponding to this period must be $2\pi a/n$.

The complete integration in terms of elliptic functions was effected by Halphen[1], and he deduced that no deformation from the circular form is possible unless the pressure Y exceed $3B/a^3$.

We shall return to this result in connexion with the theory of elastic stability.

[1] *Comptes Rendus*, XCVIII. 1884.

CHAPTER XIV.

THE BENDING AND TWISTING OF RODS OR WIRES IN THREE DIMENSIONS.

232. Kinematics of Wires[1].

THE kind of deformation experienced by a thin rod or wire which is bent or twisted depends in part on the shape assumed by the axis or elastic central-line and on the twist. Suppose that in the unstrained state the wire is straight and cylindrical or prismatic, and let any generator G be drawn upon its surface. Let P be a point on the elastic central-line, *i.e.* let P be the centre of inertia of a particular normal section, and suppose three line-elements 1, 2, 3 of the wire to proceed from P, of which (1) is in the normal section and meets the generator G, (2) is in the normal section and perpendicular to (1), and (3) is perpendicular to the normal section, *i.e.* along the elastic central-line of the wire. If P' be a point on this line near to P, line-elements drawn through P' in the same manner will be initially parallel to the line-elements 1, 2, 3. When the wire is bent and twisted these sets of line-elements no longer remain rectangular, but by means of them we can construct a system of rectangular axes of x, y, z—thus z is to be in the direction of (3), the plane (z, x) is to contain the elements (3, 1) and the axis y is to be perpendicular to this plane through the new position of P. If a system of rectangular axes be drawn in the same manner through the new position of P', the lines of this system will not in general be parallel to the corresponding axes through P, but can be derived from them by an infinitesimal

[1] Kirchhoff, *Vorlesungen über Mathematische Physik, Mechanik*, and Thomson and Tait, *Nat. Phil.* Part I. arts. 119 sq.

rotation about an axis through P. Let $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ be the components of this infinitesimal rotation about the axes x, y, z at P, then it is clear that $\delta\theta_1/PP'$ will be in the limit the curvature of the projection of the arc PP' on the plane (y, z), $\delta\theta_2/PP'$ will be the curvature of the projection on the plane (x, z), and $\delta\theta_3/PP'$ is called the *twist*.

If a rigid body move so that one point of the body describes the elastic central-line or axis of the wire in the strained position with unit velocity, and if the body rotate about this point so that three lines fixed in the body are always parallel to the axes of x, y, z, the component angular velocities of the body about these axes will be $\delta\theta_1/PP'$, $\delta\theta_2/PP'$, $\delta\theta_3/PP'$, *i.e.* they will be identical with the component curvatures[1] and twist of the wire. We shall in future denote these by κ, λ, τ respectively.

It is important to observe that in the above PP' is the length of the element of the elastic central-line after strain. In general this differs by a very small quantity from the length before strain. If ds be the length PP' before strain and ds' after strain, then

$$ds' = ds\,(1+\epsilon)\text{..........................}(1),$$

where ϵ is a very small quantity of the order of elastic strains in a solid. The quantity ϵ is the *extension* of the elastic central-line of the wire at P.

Now suppose we have constructed a system of fixed axes of ξ, η, ζ, and the system of moving axes of x, y, z whose origin is any point P of the elastic central-line of the bent and twisted wire, and let the axes of x, y, z be given with reference to the axes of ξ, η, ζ by the scheme of nine direction-cosines

$$\left.\begin{array}{cccc} & \xi & \eta & \zeta \\ x & l_1 & m_1 & n_1 \\ y & l_2 & m_2 & n_2 \\ z & l_3 & m_3 & n_3 \end{array}\right\}\text{.........................}(2),$$

where l_1, m_1, n_1 ... are direction-cosines.

The axes of x, y, z at a neighbouring point P' will be given by

[1] The curvature of a curve may be regarded as a vector directed along the binormal. The curvature of the projection on any plane through the tangent is the component of this vector along the normal to the plane.

a similar scheme in which l_1 is replaced by $l_1 + dl_1$ and so on, and we have the set of nine kinematical formulæ

$$\left.\begin{aligned} dl_1 &= l_2\delta\theta_3 - l_3\delta\theta_2, \\ dm_1 &= m_2\delta\theta_3 - m_3\delta\theta_2, \\ dn_1 &= n_2\delta\theta_3 - n_3\delta\theta_2, \\ dl_2 &= l_3\delta\theta_1 - l_1\delta\theta_3 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From these we can express κ, λ, τ in the forms

$$\left.\begin{aligned} \kappa &= \frac{\delta\theta_1}{PP'} = l_3\frac{dl_2}{ds'} + m_3\frac{dm_2}{ds'} + n_3\frac{dn_2}{ds'}, \\ \lambda &= \frac{\delta\theta_2}{PP'} = l_1\frac{dl_3}{ds'} + m_1\frac{dm_3}{ds'} + n_1\frac{dn_3}{ds'}, \\ \tau &= \frac{\delta\theta_3}{PP'} = l_2\frac{dl_1}{ds'} + m_2\frac{dm_1}{ds'} + n_2\frac{dn_1}{ds'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(4).$$

Also we have $$ds' = ds\,(1 + \epsilon),$$

where ϵ is small, so that we can replace the differential coefficients with respect to s' by differential coefficients with respect to s provided the results be multiplied by $(1 - \epsilon)$.

If the wire be unextended the correction disappears, and, if it be infinitely little bent and twisted the correction is unimportant, while if it be finitely bent and twisted the correction is again unimportant. It has been introduced here in order that the exact analytical expressions for κ, λ, τ may be obtainable from the kinematical definitions of these quantities. The correction first becomes important when the theory is extended to wires initially curved.

233. Twist.

The line-elements such as (1) are called transverses of the wire, and when the wire is bent and twisted they form a ruled surface.

In the *Natural Philosophy* of Lord Kelvin and Professor Tait, *integral twist* of any portion of a *straight* twisted wire is defined to be the angle between the transverses at its ends. If we divide the integral twist of any portion of the wire by the length of the portion we get the *average twist* of that portion, and when the portion of the wire is infinitesimal the limit of the average twist is

the *rate of twist* or simply the *twist* of the wire. Thus the twist of a straight twisted wire is the rate of rotation of the transverse about the elastic central-line per unit of length of that line.

We have already defined the twist of a bent and twisted wire by means of a rigid body exponent which moves along the strained elastic central-line with unit velocity, so that one axis fixed in the body always coincides with the tangent to that line, and one plane fixed in the body always contains the tangent to that line and the transverse. The rate of rotation of this plane about the elastic central-line per unit of its length is the twist of the wire.

Now the elastic central-line when strained forms in general a tortuous curve, and if it be simply unbent by turning each element through the angle of contingence in its osculating plane, and each osculating plane through the angle of torsion about the tangent, it will not in general be found to be untwisted. Suppose ϕ is the integral twist of a portion of the wire when its elastic central-line is thus unbent[1], so that $d\phi/ds'$ is the rate of twist of the wire when unbent. Then τ is not in general equal to $d\phi/ds'$.

Let $1/\sigma$ be the measure of tortuosity of the elastic central-line in the strained state. If the element PP' of the wire of length ds' be simply unbent it will undergo rotations $-\delta\theta_1$, $-\delta\theta_2$, and $-ds'/\sigma$ to make it a prolongation of the neighbouring element in the neighbouring osculating plane. The angle between the transverses at its ends will now be $d\phi$. It follows that to twist the wire through $d\phi$ and afterwards bend it requires a rotation about the tangent to the elastic central-line equal to $d\phi + ds'/\sigma$, and this is $\delta\theta_3$. Hence we have

$$\tau = \frac{\delta\theta_3}{ds'} = \frac{d\phi}{ds'} + \frac{1}{\sigma} \qquad\qquad (5).$$

It is geometrically obvious that if the wire be so bent that the plane through the line-elements (1) and (3) at each point becomes the osculating plane at that point ϕ will be zero, and in general we are led to expect that ϕ is the angle which the plane through these two line-elements makes with the principal normal. This may be analytically verified as follows:—

If κ, λ be the component curvatures about the axes x and y,

[1] This is the 'angular displacement' introduced into the theory by Saint-Venant. See Introduction.

the direction-cosines of the binormal of the elastic central-line at P referred to the axes of x, y, z at P are l, m, n, where

$$l = \kappa\rho, \quad m = \lambda\rho, \quad n = 0,$$

and $1/\rho$ is curvature of the elastic central-line, so that

$$1/\rho^2 = \kappa^2 + \lambda^2.$$

The direction-cosines of the binormal at P' referred to the axes at P are $l + dl$, $m + dm$, $n + dn$, where

$$\begin{aligned} dl &= d(\kappa\rho) - \lambda\rho \,.\, \tau ds', \\ dm &= d(\lambda\rho) + \kappa\rho \,.\, \tau ds', \\ dn &= \qquad\quad - \kappa\rho \,.\, \lambda ds' + \lambda\rho \,.\, \kappa ds', \end{aligned}$$

since the axes at P' are obtained from those at P by infinitesimal rotations $\kappa ds'$, $\lambda ds'$, $\tau ds'$.

Now $\qquad 1/\sigma = \{(dl)^2 + (dm)^2 + (dn)^2\}^{\frac{1}{2}}/ds'.$

Let $\kappa\rho = \sin\phi$ and $\lambda\rho = \cos\phi$, then

$$\begin{aligned} \frac{1}{\sigma^2} &= \left(\cos\phi \frac{d\phi}{ds'} - \cos\phi\tau\right)^2 + \left(-\sin\phi\frac{d\phi}{ds'} + \sin\phi\tau\right)^2 \\ &= \left(-\frac{d\phi}{ds'} + \tau\right)^2, \end{aligned}$$

so that we can have at once $\qquad \tau = \dfrac{d\phi}{ds'} + \dfrac{1}{\sigma}$,

and $\qquad \phi = \tan^{-1}(\kappa/\lambda)$......................(6).

It is scarcely necessary to remark that if the wire be twisted without bending this relation disappears.

234. General Equations of Equilibrium. Finite Displacements.

Now suppose a thin rod or wire, naturally straight, is finitely bent and twisted, and held in its new state by certain forces and couples. Let a system of moving axes of x, y, z be constructed as in art. 214, and let κ, λ, τ be the angular velocities of the moving axes about themselves as the origin moves along the axis of the wire with unit velocity. Consider the equilibrium of an element of the wire contained between two normal sections at a distance ds[1], of which the first passes through a point P on the elastic central-line and the second through a point P' on this line. Let Xds,

[1] As we are treating finite deformations we may neglect the difference between ds and ds'.

Yds, Zds be the external forces applied to the element in the directions of the axes of x, y, z at P. Let the part of the wire on the side of the normal section through P remote from P' act on the part between P and P' with a resultant force and couple whose components along and about the axes of x, y, z at P are respectively $-N_1$, $-N_2$, $-T$ for the force, and $-G_1$, $-G_2$, $-H$ for the couple,

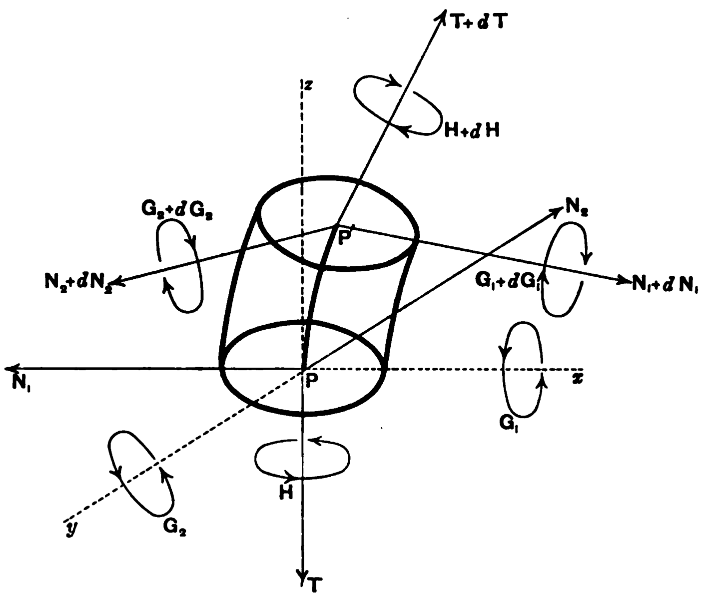


Fig. 42.

so that N_1, N_2 are the components of the shearing force and T the tension at any point, and G_1, G_2 are the components of the flexural couple and H the torsional couple. Then, remembering that the axes at P' are obtained from those at P by small rotations κds, λds, τds about the axes x, y, z at P and a translation ds along the axis z, we can write down three equations of equilibrium by resolution parallel to the axes as follows:—

The forces parallel to the axes of x, y, z at P exerted on the element PP' by the part of the wire beyond P' are respectively

$$\left.\begin{aligned} &N_1 + dN_1 - N_2\tau ds + T\lambda ds, \\ &N_2 + dN_2 - T\kappa ds \;\; + N_1\tau ds, \\ &\;\; T + dT \;\; - N_1\lambda ds + N_2\kappa ds \end{aligned}\right\} \quad \text{.................. (7).}$$

Hence the equations of resolution

$$\left.\begin{aligned}\frac{dN_1}{ds} - N_2\tau + T\lambda + X = 0,\\ \frac{dN_2}{ds} - T\kappa + N_1\tau + Y = 0,\\ \frac{dT}{ds} - N_1\lambda + N_2\kappa + Z = 0\end{aligned}\right\}\quad\text{.................(8).}$$

Again, the forces (7) act at a point whose coordinates referred to the axes of x, y, z at P are 0, 0, ds. The couples at P are $-G_1$, $-G_2$, $-H$, and the couples at P' are respectively

$$G_1 + dG_1 - G_2\tau ds + H\lambda ds,$$
$$G_2 + dG_2 - H\kappa ds + G_1\tau ds,$$
$$H + dH - G_1\lambda ds + G_2\kappa ds.$$

Hence the equations of moments about the axes at P are ultimately

$$\left.\begin{aligned}\frac{dG_1}{ds} - G_2\tau + H\lambda - N_2 = 0,\\ \frac{dG_2}{ds} - H\kappa + G_1\tau + N_1 = 0,\\ \frac{dH}{ds} - G_1\lambda + G_2\kappa \qquad = 0\end{aligned}\right\}\quad\text{.................(9).}$$

Equations (8) and (9) are the general equations of equilibrium of the wire under the action of the forces X, Y, Z.

More generally if in addition to the forces X, Y, Z couples L, M, N per unit length be directly applied to the wire the equations (9) will be replaced by

$$\left.\begin{aligned}\frac{dG_1}{ds} - G_2\tau + H\lambda - N_2 + L = 0,\\ \frac{dG_2}{ds} - H\kappa + G_1\tau + N_1 + M = 0,\\ \frac{dH}{ds} - G_1\lambda + G_2\kappa \qquad + N = 0\end{aligned}\right\}\quad\text{............(10).}$$

Equations equivalent to these were first given by Clebsch in his *Theorie der Elasticität fester Körper*.

235. The Stress-couples.

To express the flexural and torsional couples G_1, G_2, H in terms of the components of curvature κ, λ, and the twist τ we

make a particular choice of the line-elements (1) and (2) of art. 232 in terms of which these quantities were defined, viz. we assume that (1) and (2) coincide with the principal axes of inertia of the normal section at its centroid. With this choice of directions it may be shewn by an application of the general theory of Elasticity that when the rod is bent and twisted the flexural and torsional couples can be expressed in the forms

$$G_1 = A\kappa, \quad G_2 = B\lambda, \quad H = C\tau \qquad (11),$$

where A, B, C are definite constants depending on the material and the shape of the normal section. The constants A and B are called the *principal flexural rigidities* of the rod, and C the *torsional rigidity*. It can be shewn that

$$A = EI_2, \quad B = EI_1,$$

where E is the Young's modulus of the material for pull in the direction of the elastic central-line, and I_2, I_1 are the moments of inertia of the normal section at its centroid about axes respectively coinciding with the line-elements (1) and (2). The definition of C is less simple but it is a quantity of the same dimensions as A or B. For an isotropic rod of circular section it can be shewn that C is about four-fifths of A or B[1].

When the section of the rod has kinetic symmetry A and B are equal, and the line-elements (1) and (2) may be in any two directions at right angles in the normal section. In this case the two flexural couples give us a resultant couple about the binormal of the elastic central-line equal to B/ρ, where ρ is the principal radius of curvature. In the general case when A and B are unequal the resultant flexural couple is not proportional to the curvature, but its two components about principal axes are proportional to the component curvatures.

The directions of the two principal axes of inertia of the normal section and the tangent to the elastic central-line at any point P will be called the *principal torsion-flexure axes* of the rod at P.

236. Kirchhoff's Kinetic Analogue[2].

We proceed to consider the very important case where the rod is subject to terminal forces and couples only.

[1] Supposing Poisson's ratio equal to $\frac{1}{4}$ this relation is exact.

[2] *Vorlesungen über mathematische Physik, Mechanik.*

Let R be the force applied at one end P_0 of the rod, and R' the resultant of N_1, N_2, T at any point P. Then by resolving the forces that act on the piece P_0P we find that R' is equal and opposite to R, so that N_1, N_2, T are the components in the directions of the principal torsion-flexure axes at P of a force R' which is given in magnitude and direction.

If we refer to the fixed axes of ξ, η, ζ (art. 232) the equations of resolution become three such as

$$\frac{d}{ds}(l_1N_1 + l_2N_2 + l_3T) = 0,$$

which are easily reducible to the forms assumed by (8) when X, Y, Z all vanish. Hence these equations lead only to the conclusion just stated, viz. that N_1, N_2, T have a resultant which is constant in magnitude and fixed in direction.

The equations of moments (9) become by using (11)

$$\left.\begin{aligned} A\frac{d\kappa}{ds} - (B-C)\lambda\tau &= N_2, \\ B\frac{d\lambda}{ds} - (C-A)\tau\kappa &= -N_1, \\ C\frac{d\tau}{ds} - (A-B)\kappa\lambda &= 0 \end{aligned}\right\} \quad \text{..............(12)},$$

in which N_2 and $-N_1$ are equal to the moments about the axes x and y at P that the force R would have if it were applied at a point on the axis z at unit distance from P.

These equations are similar to the equations of motion of a heavy rigid body about a fixed point.

If s represent the time, and we conceive a rigid body whose moments of inertia about axes x, y, z fixed in it are A, B, C to move about the origin of x, y, z under the action of a force equal and parallel to R applied at a point on the axis z at unit distance from the origin, then κ, λ, τ given by (12) will be the component angular velocities of the rigid body about the axes of x, y, z. We may take R to be equal to the weight of the rigid body, and the direction of R to be vertical.

If a rigid body be constructed whose weight is equal to the terminal force applied to the rod, whose moments of inertia about principal axes through a certain point fixed in it are equal to the

principal flexural and torsional rigidities of the rod, and whose centre of inertia is at unit distance from the fixed point on one of the principal axes; and if the fixed point of the body be moved along the strained elastic central-line of the rod with unit velocity, and the body turn about the point under the action of gravity, the axes fixed in the body will always coincide with the principal torsion-flexure axes of the rod, and, in particular, that axis which contains the centre of inertia will always coincide with the tangent to the strained elastic central-line.

This theorem is known as Kirchhoff's *Kinetic Analogue.*

237. Terminal Couples.

Every result of the theory of the motion of a rigid body about a fixed point admits of translation into a result concerning the equilibrium of a bent and twisted wire.

Consider first the equilibrium of a wire under terminal couples[1] of which the kinetic analogue is the motion of a body under no forces. The principal torsion-flexure axes of the wire at a distance s from one end are parallel to the principal axes of the rigid body after a time s from the beginning of the motion.

The terminal couple G applied to the wire is identical with the impulsive couple required to start the rigid body or the constant angular momentum of the body. The component couples G_1, G_2, H at any point of the wire are identical with the components of angular momentum of the rigid body about the principal axes at its centre of inertia. The component curvatures and twist of the wire at any point distant s from one end are identical with the component angular velocities of the rigid body about its principal axes at time s.

If we introduce a quantity $\Theta = \sqrt{(\kappa^2 + \lambda^2 + \tau^2)}$, and regard Θ as a vector-quantity whose components parallel to the principal torsion-flexure axes at any point are the κ, λ, τ at the point, then Θ is identical with the resultant angular velocity of the rigid body, and the direction of the vector Θ is identical with that of the instantaneous axis. We may call Θ the *total torsion-flexure*

[1] Hess, 'Ueber die Biegung und Drillung eines unendlich dünnen elastischen Stabes, dessen eines Ende von einem Kräftepaar angegriffen wird'. *Math. Ann.* XXIII. 1884.

at any point, and the direction of the vector Θ the *axis of instantaneous torsion-flexure.*

We have two first integrals of the equations of equilibrium obtained from (12) by putting N_1 and N_2 equal to zero. These are

$$\left.\begin{aligned} A\kappa^2 + B\lambda^2 + C\tau^2 &= \text{const} = W,^{1} \\ A^2\kappa^2 + B^2\lambda^2 + C^2\tau^2 &= \text{const} = G^2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(13),$$

so that the potential energy per unit length and the couple across any normal section are the same at all points of the wire.

If the ellipsoid

$$\frac{A}{W}x^2 + \frac{B}{W}y^2 + \frac{C}{W}z^2 = 1,$$

roll on a fixed plane whose direction is normal to the axis of the couple G at a distance W/G from its centre, its principal axes will move so as to be parallel to the principal torsion-flexure axes of the wire. The instantaneous torsion-flexure axis at any point will be parallel to the central radius-vector to the point of contact at the corresponding instant of time, and the total torsion-flexure Θ will be equal and parallel to this radius-vector. This radius-vector traces out a cone in the ellipsoid and a cone in space. The latter intersects the plane on which the ellipsoid rolls in a curve called the *herpolhode,* the former intersects the ellipsoid in a curve called the *polhode.*

If now the cone of the instantaneous axis fixed in the ellipsoid be drawn, and the principal axes of the ellipsoid be given with reference to it, and if this cone roll on the cone whose vertex is the fixed point and base the herpolhode we shall have a representation of the motion of the rigid body.

Now suppose that at every point of the straight axis of the wire in the unstrained state a line is drawn parallel and proportional to the vector Θ that will represent in magnitude and direction the total torsion-flexure at the same point when the wire is bent and twisted by the terminal couple G, *i.e.* the projections of this line on the line-elements 1, 2, 3 of art. 232 are proportional to κ, λ, τ. These lines will form a skew surface which we may call the *surface of the polhode.* If in like manner a line be drawn from each point of the strained elastic central-line

[1] $\frac{1}{2}W$ is the potential energy per unit length of the strained wire. This will be proved later.

of the wire whose projections on the axes of x, y, z are proportional to κ, λ, τ these lines will trace out a skew surface which we may call the *surface of the herpolhode.* If the former surface (supposed flexible) be now placed in contact with the latter along a generator so that corresponding points of the axes coincide, and the successive tangent planes be made to turn about the generators so that the former surface comes to lie upon the latter, the originally straight axis of the wire will come to lie upon the bent axis in the strained state. In other words, the passage of the axis from the unstrained to the strained state may be conceived to take place by the bending of one skew surface so as to lie upon another.

Further, since the generators of the surface of the polhode are fixed in the unstrained wire, and those of the surface of the herpolhode are fixed in the bent and twisted wire, it can be easily seen that not only the axis but also every transverse of the wire will pass from its old to its new position as successive elements of the surface of the polhode are fitted to the corresponding elements of the surface of the herpolhode.

238. Symmetrical Case.

Whenever the two principal flexural rigidities A and B are equal, as in the case of a rod of uniform circular section, the third of the equations of equilibrium (12) becomes

$$\frac{d\tau}{ds} = 0,$$

so that τ is constant[1]; and the first of the integral equations (13) becomes

$$A(\kappa^2 + \lambda^2) = W - C\tau^2 = \text{const.},$$

so that $\kappa^2 + \lambda^2$ is constant and the curve of the axis of the wire is of constant curvature, while the twist of the wire is uniform.

Now the first two of the equations of equilibrium (12) become

$$A\frac{d\kappa}{ds} - (A - C)\tau\lambda = 0,$$

$$A\frac{d\lambda}{ds} + (A - C)\tau\kappa = 0.$$

[1] Poisson in his *Traité de Mécanique* found the torsional couple $C\tau$ constant. Referring to the third of equations (12) we see that in general this result requires $A = B$.

Hence $$\frac{\lambda d\kappa - \kappa d\lambda}{\lambda^2 + \kappa^2} = \frac{A - C}{A} \tau ds,$$

or $$\frac{d}{ds}\left(\tan^{-1}\frac{\kappa}{\lambda}\right) = \frac{A - C}{A}\tau.$$

Again, we have proved in art. 233 that if $1/\sigma$ be the measure of tortuosity of the elastic central-line of the wire

$$\tau = \frac{d}{ds}\left(\tan^{-1}\frac{\kappa}{\lambda}\right) + \frac{1}{\sigma},$$

hence $$\frac{1}{\sigma} = \frac{C}{A}\tau,$$

and the measure of tortuosity of the curve is constant.

Hence the elastic central-line of the wire being of constant curvature and constant tortuosity is a helix. This is the only possible form for a wire held deformed by terminal couples when the two principal flexural rigidities of the wire are equal.

The particular case of a straight wire bent into a circular arc by terminal couples can be included by taking the constant τ equal to zero.

Returning to the case of the wire bent into a helix by a couple G and fixed at one end, let α be the angle of the helix and r the radius of the cylinder on which it lies, then

$$\kappa^2 + \lambda^2 = \frac{\cos^4\alpha}{r^2},$$

$$\frac{1}{\sigma} = \frac{\cos\alpha\sin\alpha}{r},$$

so that $$\tau = \frac{A}{C}\frac{\cos\alpha\sin\alpha}{r},$$

and the couple G is given by

$$G^2 = A^2(\kappa^2 + \lambda^2) + C^2\tau^2$$
$$= A^2\frac{\cos^2\alpha}{r^2},$$

so that $$G = A\frac{\cos\alpha}{r} \qquad \text{.............................(14).}$$

Also $$\sin\alpha = \frac{C\tau}{G},$$

so that α is the angle between the plane of the applied couple and the tangent to the wire; in other words, the axis of the couple by which the wire can be held in the form of a helix when there is no terminal force is the axis of the helix.

239. Analogy to the motion of a Top[1].

The problem of the motion of a heavy rigid body aoout a fixed point has been solved in three cases, (1) when the fixed point is the centre of inertia, (2) when two principal moments of inertia at the fixed point are equal and the centre of inertia is on the third principal axis, and (3) when two principal moments are equal and each twice the third, while the centre of inertia lies in the plane perpendicular to the axis of symmetry[2]. Of these the first corresponds to the case already investigated of the equilibrium of a wire under terminal couples, and we shall proceed to consider the second. The wire must have its two principal flexural rigidities equal, and must be under the action of a force in a fixed direction applied at one end, the other end being fixed, and this force may be taken equal to the weight of the corresponding rigid body, supposed to have its centre of inertia on the axis of symmetry at a distance unity from the fixed point.

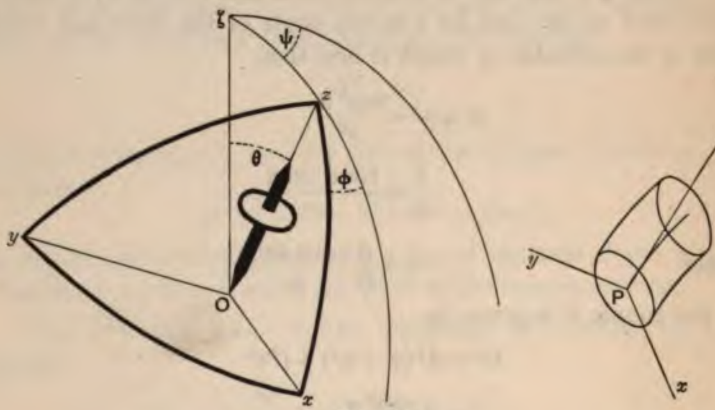


Fig. 43.

Let us take the direction of the terminal force R for the negative direction of the axis of ζ and use the Eulerian coordi-

[1] Kirchhoff, *Vorlesungen über mathematische Physik, Mechanik.*
[2] Sophie Kowalevsky, *Acta Mathematica*, XII. 1889.

nates θ, ϕ, ψ explained in Dr Routh's *Elementary Rigid Dynamics* to determine the directions of the axes of x, y, z at any point of the wire distant s from the end to which the force R is applied. Let axes be drawn from a fixed point O parallel to the axis of ζ and to the axes of x, y, z at any point of the wire. Then the forces N_1, N_2, T at any point of the wire have a resultant which is equal and opposite to R, and the rigid body in the kinetic analogue is of weight R.

The equations of equilibrium are

$$\left.\begin{aligned} A\frac{d\kappa}{ds} - \lambda\tau(A-C) &= R\cos\widehat{x\zeta} = R\sin\theta\sin\phi, \\ A\frac{d\lambda}{ds} + \kappa\tau(A-C) &= -R\cos\widehat{y\zeta} = R\sin\theta\cos\phi, \\ C\frac{d\tau}{ds} &= 0 \end{aligned}\right\}\ldots(15).$$

The geometrical equations connecting κ, λ, τ with θ, ϕ, ψ are

$$\left.\begin{aligned} \frac{d\theta}{ds} &= \kappa\sin\phi + \lambda\cos\phi, \\ \sin\theta\frac{d\psi}{ds} &= -\kappa\cos\phi + \lambda\sin\phi, \\ \frac{d\phi}{ds} + \cos\theta\frac{d\psi}{ds} &= \tau \end{aligned}\right\}\ldots\ldots\ldots\ldots(16).$$

We can write down three integrals of the equations in the forms:

$$\left.\begin{aligned} &\tau = \text{const.}, \\ &A\sin^2\theta\frac{d\psi}{ds} + C\tau\cos\theta = \text{const.}, \\ &\tfrac{1}{2}A\left\{\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\psi}{ds}\right)^2\right\} + \tfrac{1}{2}C\tau^2 + R\cos\theta = \text{const.} \end{aligned}\right\}\ldots(17),$$

of which the first follows at once from the third of (15), and the other two correspond respectively to the equations of constant moment of momentum about the vertical and constant energy in the kinetic analogue.

We shall proceed to consider the case where the motion of the Top in the kinetic analogue is steady.

240. Wire strained into a helix.

It is clear that all the equations can be satisfied by making θ and $\dfrac{d\psi}{ds}$ constants, and then we have

$$\left.\begin{aligned} \kappa\sin\phi + \lambda\cos\phi &= 0 \\ -\kappa\cos\phi + \lambda\sin\phi &= \sin\theta\,\frac{d\psi}{ds} \\ \tau &= \frac{d\phi}{ds} + \frac{d\psi}{ds}\cos\theta \end{aligned}\right\} \quad \text{.........(18).}$$

Also $\kappa^2 + \lambda^2 = \sin^2\theta\left(\dfrac{d\psi}{ds}\right)^2$, so that $\sin\theta\,\dfrac{d\psi}{ds}$ is the constant curvature of the curve assumed by the elastic central-line of the wire. This we shall denote by $1/\rho$.

Now solving the first two of equations (18) we find

$$\kappa = -\cos\phi/\rho,\quad \lambda = \sin\phi/\rho \quad \text{...............(19).}$$

hence $\cot\phi = -\kappa/\lambda$, so that the third of equations (18) becomes

$$\tau = \frac{d}{ds}\left(\tan^{-1}\frac{\kappa}{\lambda}\right) + \frac{\cot\theta}{\rho} \quad \text{..................(20).}$$

If $1/\sigma$ be the measure of tortuosity of the curve assumed by the elastic central-line of the wire we know that

$$\tau = \frac{1}{\sigma} + \frac{d}{ds}\left(\tan^{-1}\frac{\kappa}{\lambda}\right).$$

Hence $1/\sigma$ is equal to $\cot\theta/\rho$, or σ is constant, and the curvature and tortuosity of the curve assumed by the elastic central-line are both constant, so that this curve is a helix.

By equations (18) and (19) we have

$$\frac{d\kappa}{ds} = \frac{\sin\phi}{\rho}\frac{d\phi}{ds} = \lambda\left(\tau - \cos\theta\,\frac{d\psi}{ds}\right).$$

Thus the first of equations (15) becomes

$$A\lambda\left(\tau - \cos\theta\,\frac{d\psi}{ds}\right) - (A - C)\,\lambda\tau = R\sin\theta\sin\phi;$$

or by (19)

$$C\tau\sin\phi/\rho - A\cot\theta\sin\phi/\rho^2 = R\sin\theta\sin\phi.$$

Assuming that $\sin\phi$ does not vanish we deduce

$$C\tau/\rho - A\cot\theta/\rho^2 = R\sin\theta \quad \text{...............(21).}$$

Suppose now that the wire forms a helix of angle α on a cylinder of radius r, and let the terminal force and couple be R and G. We have

$$\theta = \tfrac{1}{2}\pi - \alpha, \quad 1/\rho = \cos^2\alpha/r, \quad 1/\sigma = \sin\alpha\cos\alpha/r.$$

The force R is parallel to the axis of the helix and its amount is

$$R = -A\frac{\sin\alpha\cos^2\alpha}{r^2} + C\tau\frac{\cos\alpha}{r}[1] \quad \text{............(22)}.$$

The couple G has components $A\cos^2\alpha/r$ about the binormal and $C\tau$ about the tangent at the end of the wire.

If the wire be attached to rigid terminals to which forces and couples are applied we may suppose the force and couple reduced to a wrench applied to the terminal. The force of the wrench is R, and the couple K is the component of G about the line of action of R, so that

$$K = A\cos^3\alpha/r + C\tau\sin\alpha \quad \text{..................(23)}.$$

The axis of the wrench is parallel to the axis of the helix, and one point on it can be found by drawing a line (the principal normal at the end) perpendicular to the plane of R and G, and marking off upon it a point O such that the moment of R about O is equal to the other component of G. The distance of this point from the end of the wire is

$$[-\sin\alpha\, A\cos^2\alpha/r + \cos\alpha\, C\tau] \div R$$

which is r, so that the axis of the wrench coincides with the axis of the helix.

A given wire of equal flexural rigidity in all planes through its elastic central-line can be held in the form of a given helix with a given twist by the application of equal and opposite wrenches about the axis of the helix to rigid terminals. The force R and the couple K of either wrench are given by the equations (22) and (23).

If the wire be held by couples only we must have

$$\tau = \frac{A}{C}\frac{\sin\alpha\cos\alpha}{r}$$

[1] The signs have been so chosen that R is positive when it tends to pull out the wire in the direction of the axis of the helix.

as in art. 238, but if the wire be held by forces only we must have

$$\tau = -\frac{A}{C}\frac{\cos^2\alpha}{\sin\alpha}\frac{1}{r};$$

so that the same wire can be held in the same helix by force alone, or by couple alone, or by a wrench of given pitch, but in each case the twist must be properly adjusted.

241. Problems on the Equilibrium of Wires.

The problem of determining the forces by which a given wire can be held in the form of a given curve can be investigated from the general equations (8) and (9) of art. 234, but in general it will be found impossible to hold the wire by terminal forces and couples alone.

From the third of equations (9) it appears that when no couples are directly applied except at the ends the two component curvatures and the twist cannot be given arbitrarily. Suppose the curve given so that the measure of curvature $1/\rho$ and the measure of tortuosity $1/\sigma$ are given functions of s, and let $\kappa = \sin\phi/\rho$ and $\lambda = \cos\phi/\rho$. Then the third of equations (9) becomes

$$C\frac{d}{ds}\left(\frac{d\phi}{ds} + \frac{1}{\sigma}\right) = (A - B)\frac{\sin\phi\cos\phi}{\rho^2} \quad \text{.........(24)}$$

an equation to determine ϕ as a function of s. We may express this by saying that a given wire cannot be held in the form of a given curve unless either a certain torsional couple be applied to it directly at each section, or the wire have a certain amount of twist. This twist is indeterminate to the extent of an arbitrary constant, so that if a wire can be held in a given curve with a certain twist it can still be so held when the twist is increased uniformly.

Suppose now that the wire is suitably twisted and held in the form of the given curve, the components N_1 and N_2 of the shearing force are given by the first two of equations (9). We have then three relations (8) connecting the four quantities T, X, Y, Z. We may therefore impose one condition which these forces must satisfy; *e.g.* we may find the forces in order that the tension T may be a given constant, or we may suppose the wire to be acted upon by purely normal forces so that Z vanishes. (Cf. art. 230.)

As an example of a plane curve with uniform twist we may suppose a straight wire uniformly twisted and bent into an arc of a circle. It may be easily shewn that the wire can be in equilibrium in the form of an arc of a circle under terminal forces and couples, or in the form of a complete circle, and there will be a thrust T across each section equal to $(B-C)\tau^2$, where B and C are the flexural and torsional rigidities, and τ the twist. The shearing force at any section is perpendicular to the plane of the circle, and equal to $(B-C)\tau/\rho$, where ρ is the radius of the circle.

As an example[1] of equations (10) of art. 234 we may suppose a wire bent into a circle without twisting so as to have one of the principal torsion-flexure axes in any section directed along the normal to the circle, and then suppose that each normal section is turned through an angle ϕ in its own plane round the elastic central-line. Technically speaking this is not twisting the wire but readjusting the plane of flexure with reference to the material.

Let r be the radius of the circle,

then $$\lambda = \cos\phi/r, \quad \kappa = \sin\phi/r, \quad \tau = 0.$$

The couples L and M will not be required, and we may put them each equal to zero, but the couple N will be given by

$$(A-B)\sin\phi\cos\phi/r^2 = N \quad \ldots\ldots\ldots\ldots\ldots(25).$$

No couple will be necessary if the sections be turned through a right angle or through two right angles, or if the two principal flexural rigidities be equal.

242. Theory of Wires initially curved.

The theory of naturally straight prismatic wires may be generalised so as to cover the case where the wire in its natural state is curved, and is such that if its elastic central-line were transformed into a straight line by bending through the angle of contingence in each osculating plane and by turning each osculating plane through the angle of torsion about the tangent it would not be prismatic. We shall consider a wire whose normal sections are of the same shape throughout, and we shall suppose that in the natural state its elastic central-line forms a curve

[1] Thomson and Tait, *Nat. Phil.* Part II. art. 623.

whose principal curvature at any point is $1/\rho$ and whose measure of tortuosity is $1/\sigma$. We shall further suppose that in the natural state one of the principal axes of inertia of the normal section at its centroid makes with the principal normal to the elastic central-line an angle ϕ_0. Then ϕ_0 is a variable which is supposed to be given at every point of the curve, and we shall assume that ϕ_0 varies continuously according to some assigned law.

In the natural state let P be any point on the elastic central-line of the wire, and let three line-elements of the wire 1, 2, 3 proceed from P, of which (3) lies along the tangent to the elastic central-line at P, and (1) and (2) lie along the principal axes of inertia of the normal section through P. Then at every point the line-elements drawn in this way are at right angles. We shall suppose that the line-element (1) makes an angle ϕ_0 with the principal normal to the elastic central-line of the wire at P. The system of line-elements forms a system of moving axes, and we shall suppose that the line-elements drawn through a neighbouring point P' distant ds from P can be obtained from those at P by infinitesimal rotations κds, λds, τds about the axes at P. Then κ and λ are the component curvatures of the elastic central-line of the wire about the line-elements (1) and (2), and ϕ_0 differs from $\tan^{-1}(\kappa/\lambda)$ by a constant, so that we have

$$\tau = \frac{d\phi_0}{ds} + \frac{1}{\sigma} = \frac{d}{ds}\left(\tan^{-1}\frac{\kappa}{\lambda}\right) + \frac{1}{\sigma} \quad \text{............(26)}.$$

Now when the wire is further bent and twisted the line-elements 1, 2, 3 do not remain rectangular, but by means of them we can construct a system of rectangular axes of x, y, z to which we can refer points in the neighbourhood of P. Thus P is to be the origin, the line-element (3) is to lie along the axis z, and the plane through the line-elements (1) and (3) is to contain the axes of x and z.

Suppose now that the axes of x, y, z at P' drawn as above described could be obtained from those at P by elementary rotations $\kappa' ds'$, $\lambda' ds'$, $\tau' ds'$ about the axes at P, where ds' is the length of PP' after strain, then κ' and λ' are the component curvatures of the strained elastic central-line of the wire, and $\tau' - \tau$ measures the *twist*.

The originally curved wire could be held straight and prismatic, and there would be at every section couples $-A\kappa$, $-B\lambda$, $-C\tau$

about the line-elements 1, 2, 3, where A and B are the two principal flexural rigidities, and C is the torsional rigidity. These quantities depend only on the size and shape of the normal section and on the elastic constants. The straight prismatic wire could be transformed into the bent and twisted wire by the application at every section of additional couples $A\kappa'$, $B\lambda'$, $C\tau'$. It follows that when the naturally curved wire is bent and twisted the stress-couples at every normal section about the axes x, y, z are $A(\kappa'-\kappa)$, $B(\lambda'-\lambda)$, $C(\tau'-\tau)$[1].

It is to be noticed that the resultant flexural couple cannot be proportional to the change of curvature unless $A=B$, or the wire is of equal flexural rigidity in all planes through its elastic central-line, and the couple will not be proportional to the change of curvature, even when $A=B$, unless $\kappa\lambda'=\lambda'\kappa$, *i.e.* unless the angle between a principal axis of inertia of the normal section at any point and the principal normal to the elastic central-line is unaltered by the strain. In other words, the necessary condition is that the same set of transverses of the wire which were initially principal normals are principal normals after strain. If this condition be satisfied the twist is equal to the difference of the measures of tortuosity of the curve assumed by the elastic central-line before and after strain.

243. Equations of Equilibrium.

We are going to treat the finite deformation of naturally curved wires. As κ', λ', and τ' are finite we may now neglect the difference between ds and ds', or regard the elements of the elastic central-line as unextended.

Let G_1, G_2, and H be the components referred to the axes of x, y, z of the flexural couple and the torsional couple at any section, N_1, N_2, and T the components of the shearing force and the tension referred to the same axes. Then, remembering that $\kappa' ds$, $\lambda' ds$, $\tau' ds$ are the component rotations executed by the axes about themselves as we pass from point to point of the strained elastic central-line of the wire, we see as in art. 234 that the equations of equilibrium under forces X, Y, Z per unit length parallel to the axes of x, y, z can be expressed in the forms:—

[1] Clebsch, *Theorie der Elasticität fester Körper.*

three equations of resolution

$$\left.\begin{aligned} &\frac{dN_1}{ds} - N_2\tau' + T\lambda' + X = 0, \\ &\frac{dN_2}{ds} - T\kappa' + N_1\tau' + Y = 0, \\ &\frac{dT}{ds} - N_1\lambda' + N_2\kappa' + Z = 0 \end{aligned}\right\} \text{...............(27);}$$

and three equations of moments

$$\left.\begin{aligned} &\frac{dG_1}{ds} - G_2\tau' + H\lambda' - N_2 = 0, \\ &\frac{dG_2}{ds} - H\kappa' + G_1\tau' + N_1 = 0, \\ &\frac{dH}{ds} - G_1\lambda' + G_2\kappa' \qquad = 0 \end{aligned}\right\} \text{...............(28).}$$

These are the general equations of equilibrium of the wire under forces X, Y, Z per unit length. If there be also couples L, M, N per unit length we shall have to add L, M, N to the left-hand sides of (28).

In these equations

$$\left.\begin{aligned} G_1 &= A(\kappa' - \kappa), \\ G_2 &= B(\lambda' - \lambda), \\ H &= C(\tau' - \tau) \end{aligned}\right\} \text{.....................(29).}$$

Equations equivalent to those given in this article were obtained by Clebsch in his *Theorie der Elasticität fester Körper*.

In the case where the wire is under the action of terminal forces and couples only we may simplify the equations (27) by omitting X, Y, and Z, and then these equations shew, as in art. 236, that the force whose components are N_1, N_2, T at any section is equal and opposite to the force R applied at an end.

244. Extension of Kirchhoff's Kinetic Analogue.

Mr Larmor[1] has given an interesting extension of Kirchhoff's theorem of the kinetic analogue to the case of a wire initially helical and without twist.

Suppose that the natural form of the elastic central-line of the wire is a helix, and that if the wire were simply unbent into a

[1] *Proc. Lond. Math. Soc.* xv. 1884.

straight rod its surface would be prismatic, then κ, λ, and τ are constants.

If we write

$$\left.\begin{matrix} A(\kappa'-\kappa)=h_1, & B(\lambda'-\lambda)=h_2, & C(\tau'-\tau)=h_3, \\ \kappa'=\theta_1, & \lambda'=\theta_2, & \tau'=\theta_3 \end{matrix}\right\}\ldots(30),$$

and suppose κ, λ, τ constant, equations (28) can be transformed into

$$\left.\begin{aligned} \frac{dh_1}{ds}-h_2\theta_3+h_3\theta_2 &= N_2, \\ \frac{dh_2}{ds}-h_3\theta_1+h_1\theta_3 &= -N_1, \\ \frac{dh_3}{ds}-h_1\theta_2+h_2\theta_1 &= 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(31).$$

If we could find a dynamical system whose moments of momentum referred to a determinate set of axes rotating about a fixed origin with angular velocities θ_1, θ_2, θ_3 were h_1, h_2, h_3, then equations (31) would be the equations of motion of the dynamical system under the action of the force R, applied at a point on the axis (3) at unit distance from the fixed origin, and the system would move so that the axes 1, 2, 3 would be parallel at time s to the principal torsion-flexure axes of the bent and twisted wire at an arc-distance s from the end of the elastic central-line of the wire at which the force R is applied.

The required dynamical system can be constructed by mounting a rigid fly-wheel on an axis fixed in a solid which rotates about a fixed point, the fly-wheel being free to rotate about the axis.

Suppose that when the fly-wheel and the rigid body move as if rigid the moments of inertia of the rigid body and fly-wheel about principal axes through the fixed point are A, B, C. Let the centre of inertia of the fly-wheel be at the fixed point, and let it rotate about its axis with constant angular momentum h, and suppose the axis of the fly-wheel to make with the principal axes of the rigid body and fly-wheel (whose position is unaltered by the rotation) angles whose cosines are l, m, n. Then if we take

$$-hl=A\kappa, \quad -hm=B\lambda, \quad -hn=C\tau$$

h and l, m, n will be determinate, and the moments of momentum of the rigid body and fly-wheel about the fixed point will be h_1, h_2, h_3.

If now the rigid body and fly-wheel be of weight R, and a point on one of the principal axes at its centre of inertia at unit distance from that centre be the fixed point, the body will move under gravity so that its principal axes at the fixed point at time s are parallel to the principal torsion-flexure axes of the wire at an arc-distance s from a fixed point on its elastic central-line.

245. Particular Cases.

The above includes the particular case of a circular wire bent in its own plane by terminal forces and couples. The kinetic analogue is the motion of a pendulum about a fixed horizontal axis on which a fly-wheel is symmetrically mounted. It is clear that the motion of the pendulum will not be affected by that of the fly-wheel, and that the form assumed by the naturally circular wire must therefore be one of the forms of the elastica. There is no difficulty in verifying this from the equations of equilibrium of the wire. The only way in which the fact that the wire is naturally circular affects the problem is that the terminal couple required to hold the wire in the form of a given elastica is diminished by the terminal couple that would be required to bend a naturally straight wire into the given originally circular form.

Another particular case is that in which the wire is naturally straight and of uniform section but appears uniformly twisted, *i.e.* where κ and λ vanish while τ is constant. We shall consider especially the case where the wire is of uniform flexibility in all planes through its elastic central-line. The condition that τ is constant means that the normal sections are not similarly situated but homologous lines in them form a helicoid of given pitch. In this case the h_1, h_2 of art. 244 become $A\theta_1, A\theta_2$, but h_3 is not $C\theta_3$. The equations become identical with the equations of motion of an ordinary gyrostat or top referred to its axis of figure and to two perpendicular axes not fixed in the top. The steady motion of the top will correspond, as we shall see, to a helical form of the bent and twisted wire.

The equations of equilibrium (28) become

$$\left.\begin{aligned} A\frac{d\kappa'}{ds} - A\lambda'\tau' + C(\tau'-\tau)\lambda' &= \quad N_2, \\ A\frac{d\lambda'}{ds} - C(\tau'-\tau)\kappa' + A\kappa'\tau' &= -N_1, \\ C\frac{d}{ds}(\tau'-\tau) \qquad\qquad &= 0 \end{aligned}\right\} \text{......(32)},$$

in which N_1, N_2 are the components of the force R reversed parallel to the axes of x and y, while κ' and λ' are the component curvatures about the same axes.

With the notation of art. 239 we see that, when the elastic central-line makes a constant angle θ with the line of action of the force R reversed,

$$N_2 = R\sin\theta\sin\phi, \qquad -N_1 = R\sin\theta\cos\phi,$$
$$\kappa' = -\cos\phi/\rho, \qquad \lambda' = \sin\phi/\rho,$$

where ρ is the radius of curvature of the strained elastic central-line, so that $1/\rho^2 = \kappa'^2 + \lambda'^2$.

Hence the equations

$$\left.\begin{aligned} N_2\kappa' - N_1\lambda' &= 0, \\ N_2\lambda' + N_1\kappa' &= R\sin\theta/\rho \end{aligned}\right\} \text{..................(33)}.$$

Take equations (32), multiply the first two of them by κ' and λ' respectively, and add. We find by the first of (33)

$$\kappa'\frac{d\kappa'}{ds} + \lambda'\frac{d\lambda'}{ds} = 0, \text{ or } \kappa'^2 + \lambda'^2 = \text{const.} = 1/\rho^2 \text{ ......(34)}.$$

Again, multiply the first and second of equations (32) by λ' and $-\kappa'$ and add. We find by the second of (33)

$$A\left(\lambda'\frac{d\kappa'}{ds} - \kappa'\frac{d\lambda'}{ds}\right) - A\tau'(\lambda'^2 + \kappa'^2) + C(\tau'-\tau)(\lambda'^2 + \kappa'^2) = R\sin\theta/\rho$$

or, dividing by $(\kappa'^2 + \lambda'^2)$, which is equivalent to multiplying by ρ^2, we have

$$A\frac{d}{ds}\left(\tan^{-1}\frac{\kappa'}{\lambda'}\right) - A\tau' + C(\tau'-\tau) = R\rho\sin\theta \text{ ......(35)}.$$

Again, the third of equations (32) gives

$$\tau' - \tau = \text{const. ....(36)}.$$

Now by supposition τ is constant so that τ' is constant, and therefore by (34) and (35) it follows that $\frac{d}{ds}\left(\tan^{-1}\frac{\kappa'}{\lambda'}\right)$ is constant.

6—2

This quantity is identical with $\tau' - 1/\sigma$, where $1/\sigma$ is the measure of tortuosity of the strained elastic central-line. Hence this line is of constant curvature and tortuosity, and it is therefore a helix.

It is now easy to shew that if α and r be the angle and radius of the helix, the terminal force and couples are:
a force R given by

$$R = C(\tau' - \tau)\frac{\cos\alpha}{r} - A\frac{\sin\alpha\cos^2\alpha}{r^2}\text{...........}(37),$$

a couple K about a line parallel to the axis of the helix given by

$$K = C(\tau' - \tau)\sin\alpha + A\cos^3\alpha/r \text{ ....}(38),$$

and a couple Rr about the tangent at the end of the wire to the circular section of the cylinder on which the helix lies.

If, as in art. 240, the wire be attached to rigid terminals the system of forces applied to the terminal reduces to a wrench (R, K) about the axis of the helix.

The results of that article apply equally therefore to a prismatic wire and to one in which homologous lines in the normal sections form a helicoid of given pitch, provided the sections have kinetic symmetry.

246. Helical wire bent into new helix.

Consider the case of a wire of equal flexibility in all planes through its axis, and suppose the initial form is a helix, and that if the wire were simply unbent it would be straight and prismatic. Then we may take $\kappa = 0$, $\lambda = 1/\rho$, $\tau = 1/\sigma$ where $1/\rho$ and $1/\sigma$ are the principal curvature and the measure of tortuosity of the initial helix. It is clear that a possible final state of the wire will be given by $\kappa' = 0$, $\lambda' = 1/\rho'$, $\tau' = 1/\sigma'$, where ρ' and σ' are constants, *i.e.* the wire is simply bent into a new helix.

The first two of the equations of equilibrium (28) become

$$N_2 = -A\left(\frac{1}{\rho'} - \frac{1}{\rho}\right)\frac{1}{\sigma'} + C\left(\frac{1}{\sigma'} - \frac{1}{\sigma}\right)\frac{1}{\rho'},$$
$$N_1 = 0,$$

while the third is identically satisfied.

Again, the first two of equations (27) with X, Y, and Z put equal to zero are identically satisfied, while the third gives us

$$T/\rho' = N_2/\sigma',$$

or $T = N_2 \tan\alpha'$, if α' be the angle of the new helix. This condition expresses that the terminal force is parallel to the axis of the helix.

It can now be shewn without difficulty that the forces and couples at a rigid terminal reduce to a wrench (R, K) whose axis coincides with the axis of the helix, and whose components of force R and couple K are given by

$$\left.\begin{aligned} R &= \frac{C\cos\alpha'}{r'}\left\{\frac{\sin\alpha'\cos\alpha'}{r'} - \frac{\sin\alpha\cos\alpha}{r}\right\} - \frac{A\sin\alpha'}{r'}\left\{\frac{\cos^2\alpha'}{r'} - \frac{\cos^2\alpha}{r}\right\}, \\ K &= C\sin\alpha'\left\{\frac{\sin\alpha'\cos\alpha'}{r'} - \frac{\sin\alpha\cos\alpha}{r}\right\} + A\cos\alpha'\left\{\frac{\cos^2\alpha'}{r'} - \frac{\cos^2\alpha}{r}\right\} \end{aligned}\right\} \quad \text{............(39)},$$

where α' and r' are the angle and radius of the helix into which the wire is strained, and α and r are corresponding quantities in the natural state.

247. Spiral Springs [1].

We can hence obtain a theory of the small deformation of spiral springs, by taking as variables the length of the axis between the terminals and the angle through which one extremity rotates about the axis.

If l be the length of the elastic central-line and z the length between the terminals, and if

$$z = l\sin\alpha, \quad \phi = l\cos\alpha/r;$$

then

$$z + \delta z = l\sin\alpha', \quad \phi + \delta\phi = l\cos\alpha'/r'$$

will be the values of z and ϕ in the new spiral.

We get

$$l\cos\alpha = \sqrt{(l^2 - z^2)},$$

$$l\cos\alpha' = \sqrt{\{l^2 - (z+\delta z)^2\}}.$$

Hence

$$1/r = \phi/\sqrt{(l^2 - z^2)},$$

$$1/r' = (\phi + \delta\phi)/\sqrt{\{l^2 - (z+\delta z)^2\}}.$$

In case $\delta\phi$ and δz are very small we get approximately

$$l\cos\alpha' = l\cos\alpha - \tan\alpha\,\delta z,$$

$$1/r' = 1/r + \frac{\delta\phi}{l\cos\alpha} + \frac{\phi\,\delta z\sin\alpha}{l^2\cos^3\alpha}.$$

[1] Thomson and Tait, *Nat. Phil.* Part II. arts. 604—607.

Hence $$\frac{\cos^2\alpha'}{r'} - \frac{\cos^2\alpha}{r} = \cos\alpha\frac{\delta\phi}{l} - \phi\frac{\delta z}{l^2}\tan\alpha,$$

$$\frac{\sin\alpha'\cos\alpha'}{r'} - \frac{\sin\alpha\cos\alpha}{r} = \sin\alpha\frac{\delta\phi}{l} + \phi\frac{\delta z}{l^2}.$$

Or $$\frac{\cos^2\alpha'}{r'} - \frac{\cos^2\alpha}{r} = \frac{\sqrt{(l^2-z^2)}}{l^2}\delta\phi - \phi\frac{z\delta z}{l^2\sqrt{(l^2-z^2)}},$$

$$\frac{\sin\alpha'\cos\alpha'}{r'} - \frac{\sin\alpha\cos\alpha}{r} = \frac{z\delta\phi}{l^2} + \frac{\phi\delta z}{l^2}.$$

Hence we should find from (39) that the force and couple at the terminals are

$$\left.\begin{aligned} R &= \frac{1}{l^3}\left[\left(C + A\,\frac{z^2}{l^2-z^2}\right)\phi^2\delta z + (C-A)\,\phi z\delta\phi\right], \\ K &= \frac{1}{l^3}\left[(C-A)\,\phi z\delta z + \{A\,(l^2-z^2) + Cz^2\}\,\delta\phi\right] \end{aligned}\right\} \quad \ldots\ldots (40).$$

If the couple be put equal to zero we have the value of $\delta\phi$ in terms of δz and can hence find the force required to produce a given axial displacement δz. If on the other hand $\delta\phi$ be put equal to zero we can find the force and couple required to produce a given axial displacement when the terminal is constrained not to rotate.

248. Problems on the Equilibrium of Wires[1].

As an example of the application of the equations of equilibrium when couples are applied directly to the wire we may consider the following problem:—A wire, which when unstrained is in the form of a circular arc of radius a with one principal torsion-flexure axis of each section along the normal to the circle, is simply bent into a circular hoop of radius r, and then each section is turned through an angle ϕ about the elastic central-line; it is required to find the torsional couple that must be applied to hold the wire.

The initial state is expressed by

$$\kappa = 0, \quad \lambda = 1/a, \quad \tau = 0,$$

and the final state is expressed by

$$\kappa' = \sin\phi/r, \quad \lambda' = \cos\phi/r, \quad \tau' = 0.$$

[1] Thomson and Tait, *Nat. Phil.* Part II. arts. 624, 625.

The equation determining the torsional couple N is

$$A(\kappa' - \kappa)\lambda' - B(\lambda' - \lambda)\kappa' = N,$$

or
$$N = B\frac{\sin\phi}{ar} + (A - B)\frac{\sin\phi\cos\phi}{r^2} \qquad \text{............(41).}$$

(Cf. art. 241.)

As a further example suppose that when the wire forms a circular arc of radius a the principal torsion-flexure axis for which the flexural rigidity is A is not in the plane of the circle but inclined to it at an angle ϕ_0. Suppose that the wire is bent into a circular hoop, and that the same axis in the strained state is inclined to the plane of the hoop at an angle ϕ, it is required to find ϕ so that the hoop may be in equilibrium without any applied couple.

The initial state is expressed by

$$\kappa = \sin\phi_0/a, \quad \lambda = \cos\phi_0/a, \quad \tau = 0,$$

and the final state is expressed by

$$\kappa' = \sin\phi/r, \quad \lambda' = \cos\phi/r, \quad \tau' = 0.$$

The condition that the hoop may be in equilibrium without any applied couple is by the third of equations (28)

$$-A(\kappa' - \kappa)\lambda' + B(\lambda' - \lambda)\kappa' = 0,$$

or
$$A\frac{\cos\phi}{r}\left(\frac{\sin\phi}{r} - \frac{\sin\phi_0}{a}\right) = B\frac{\sin\phi}{r}\left(\frac{\cos\phi}{r} - \frac{\cos\phi_0}{a}\right) \text{...(42).}$$

CHAPTER XV.

GENERAL THEORY OF THIN RODS OR WIRES.

249. Basis of Kirchhoff's Theory.

The modern theory of thin rods or wires, as also that of thin plates, was founded by Kirchhoff, starting from the remark that, when different linear dimensions of a body are of different orders of magnitude, the general equations of equilibrium or small motion given by the theory of Elasticity apply, in the first instance, not to the whole body but to a small part of it, whose linear dimensions are all of the same order of magnitude. In the general equations of Elasticity infinitesimal displacements are alone regarded, while a little observation shews that a very thin rod or plate may undergo a considerable change of shape without taking any set, and therefore without experiencing more than a very small strain. To investigate the changes of shape of such a body by means of the elastic equations it is necessary to suppose it divided into a very large number of small bodies whose linear dimensions are all of the same order of magnitude, and to consider the small relative displacements of the parts of each such small body. For example, in the case of a straight cylindrical rod whose length is great compared with the greatest diameter of its cross-section, it is necessary to consider the rod as divided into elementary prisms each of length comparable with this diameter. If the length of the rod be regarded as of the same order of magnitude as the unit of length, we may say simply that each of the linear dimensions of one of these prisms is small, and speak of these prisms as *elementary*. Kirchhoff's theory is an approximate one obtained by regarding the elementary prism as infinitesimal.

With regard to the equilibrium of an elementary prism of a rod, or of any small body whose linear dimensions are all of the same order of magnitude and small compared with the unit of length, we can state at once the principle which forms the foundation of Kirchhoff's method[1], viz. *the internal strain in such an element depends on the surface-tractions on its faces and not on the bodily forces nor on the kinetic reactions arising from its inertia.* The argument in I. art. 13 by which, from the consideration of an elementary tetrahedron, a relation was found among surface-stresses applies equally to any small body or element of a body such as those considered in Kirchhoff's theory, and shews that when the dimensions of the body or element are infinitesimal the principle above stated is exact. Kirchhoff's proof of the principle proceeds thus:—

The surface-conditions[2] such as

$$lP + mU + nT = F \qquad\qquad (1),$$

shew that the parts of the stresses due to the surface-tractions are of the same order as those tractions, while the equations such as

$$\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} + \rho X = \rho \frac{\partial^2 u}{\partial t^2} \qquad\qquad (2),$$

shew that the parts of the stresses due to the bodily forces and kinetic reactions are of a higher order than those forces in the small quantities x, y, z, which are the coordinates of a point in the elementary body referred to axes fixed in it.

The principle shews that for a first approximation we may reduce the equations determining the stress within the element to the forms

$$\left.\begin{aligned} \frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial T}{\partial z} = 0, \\ \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial S}{\partial z} = 0, \\ \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial R}{\partial z} = 0 \end{aligned}\right\} \qquad\qquad (3),$$

and proceed to solve these subject to the boundary-conditions.

A second principle of equal importance is that of the *Elastic equivalence of statically equipollent systems of load.* This has

[1] Crelle LVI. 1860, and *Vorlesungen über mathematische Physik, Mechanik.*

[2] See I. art. 15.

already been fully explained in I. arts. 107 and 152. The application of it to the theory of thin rods will be found in the statement that it can make very little difference to the form assumed by the elastic central-line of the rod how the forces that deform it are distributed over an element, provided they give rise to the same resultants and moments, and we may therefore regard the forces applied to an elementary prism from without as replaced by their force- and couple-resultants at the centre of the prism. The same will hold of the stresses across the normal sections of the rod that arise from the action of contiguous prisms. A particular deduction from this principle, of great importance, is that we may always regard the cylindrical or tubular bounding-surface of the rod as free from stress. When the deformation is produced by tractions actually applied to this surface, they may, without affecting the results, be replaced by a bodily distribution of force and couple acting through the substance of each elementary prism, the only condition being that the actual distribution and that which replaces it must be statically equivalent.

250. Explanation of the method.

We are now in a position to explain the method by which the values of the stress-couples G_1, G_2, H of art. 235 can be calculated. Referring to art. 232 we see that the form of a bent and twisted rod depends upon three quantities, κ, λ, τ, which define the curvature of its elastic central-line and the twist. The deformation of any elementary prism will clearly be in part dependent on the same quantities, and certain relations must obtain among the relative displacements of the particles of such a prism in order that the various prisms may after strain continue to form a continuous rod of curvature and twist expressed by κ, λ, τ. The first step is the investigation of these relations; we shall call them Kirchhoff's *differential identities*. In virtue of these relations the strain in an elementary prism will be partly determined, and the second step is the establishment of a method by which approximate values of the components of strain can be deduced. We shall find that the first approximation can be very easily carried out, and that, for the purpose of finding the values of the stress-couples, it is unnecessary to proceed to a second approximation.

We recall here the notation of art. 232. We there supposed

P to be any point of the elastic central-line of the rod, and that three line-elements (1, 2, 3) of the wire proceed from P. We shall take the line-elements (1) and (2) to be the principal axes at P of the normal section through P, and (3) the tangent to the elastic central-line at P. By means of these three line-elements we constructed a system of moving rectangular axes of x, y, z, having the origin at the strained position of P, the axis z along the tangent to the strained elastic central-line at P, and the plane (x, z) through the line-elements (1) and (3). When the rod is deformed any element PP' of the elastic central-line whose unstrained length is ds will become an element of the strained elastic central-line whose length is ds', where

$$ds' = ds\,(1 + \epsilon) \qquad\qquad (4),$$

ϵ being the extension of the elastic central-line at P. The quantity ϵ can be of the order of magnitude of small strains in an elastic solid. The axes of x, y, z at P' constructed in the same manner as those at P will not be parallel to the axes at P after strain, but can be derived from them by infinitesimal rotations $\kappa ds'$, $\lambda ds'$, $\tau ds'$ about the axes of x, y, z at P. The quantities κ, λ are the component curvatures, and τ the twist of the rod at P.

251. Kirchhoff's Identities.

Let Q be the position of any particle of the rod near to P, and let the coordinates of Q referred to the line-elements at P before strain be x, y, z; after strain, referred to the axes of x, y, z let them be $x+u$, $y+v$, $z+w$. Then u, v, w are the displacements of Q, and if s be the distance of P from a fixed point on the elastic central-line u, v, w are functions of x, y, z and s.

Suppose P' is a point on the elastic central-line near to P, then we might refer Q to P' instead of P. The coordinates of Q before strain referred to P' will be x, y, $z-ds$, where ds is the unstrained length of PP'. To obtain the values of the displacements of Q referred to P' from the values of the displacements of Q referred to P we must in u, v, w replace s by $s+ds$ and z by $z-ds$. It follows that the coordinates of Q referred to P' after strain will be

$$x + u + \frac{\partial u}{\partial s}ds - \frac{\partial u}{\partial z}ds,\; y + v + \frac{\partial v}{\partial s}ds - \frac{\partial v}{\partial z}ds,$$

$$z - ds + w + \frac{\partial w}{\partial s}ds - \frac{\partial w}{\partial z}ds \qquad\qquad (5).$$

Now since the axes at P' are obtained from those at P by a translation $ds' = (1+\epsilon)\,ds$ along the axis z, and rotations $\kappa ds'$, $\lambda ds'$, $\tau ds'$ about the axes x, y, z, the coordinates of Q referred to P' are

$$\left.\begin{array}{l} x+u+(y+v)\,\tau ds'-(z+w)\,\lambda ds', \\ y+v+(z+w)\,\kappa ds'-(x+u)\,\tau ds', \\ z+w+(x+u)\,\lambda ds'-(y+v)\,\kappa ds'-ds' \end{array}\right\} \text{.........(6).}$$

Hence comparing the expressions (5) and (6) for the coordinates of Q referred to P', and remembering that $ds' = ds\,(1+\epsilon)$, we find

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial z}=\dfrac{\partial u}{\partial s}-\{\tau(y+v)-\lambda(z+w)\}(1+\epsilon), \\ \dfrac{\partial v}{\partial z}=\dfrac{\partial v}{\partial s}-\{\kappa(z+w)-\tau(x+u)\}(1+\epsilon), \\ \dfrac{\partial w}{\partial z}=\dfrac{\partial w}{\partial s}-\{\lambda(x+u)-\kappa(y+v)\}(1+\epsilon)+\epsilon \end{array}\right\} \text{...... (7).}$$

These are Kirchhoff's differential identities and the above method of proving them is due to M. Boussinesq[1].

252. Method of approximation.

By these equations the displacements u, v, w become to a certain extent determinate, and if the equations could be solved we should obtain a certain amount of information in regard to the possible forms of the displacements. We proceed to indicate a method of solution by successive approximation depending on the fact that x, y, z are everywhere small, and u, v, w are small in comparison with x, y, z, while ϵ is a small quantity which is at most of the order of strains in an elastic solid.

There are three classes of cases to be considered. In the first some at least of the quantities κ, λ, τ defining the curvature and twist of the rod may be finite. In this case we may reject terms in ϵ as small in comparison with terms in κx, κy where κ is taken to stand for any one of the quantities κ, λ, τ which is finite. There will exist modes of infinitesimal deformation continuous with these for which κ, λ, τ are very small and ϵ so small as to be negligible in comparison with terms of the order of the product $\kappa \times$ (diameter of normal section). In the second class of cases terms of the order κx are negligible in comparison with ϵ. This

[1] 'Étude nouvelle sur l'équilibre...des solides...dont certaines dimensions sont très petites...'. Liouville's *Journal*, XVI. 1871.

corresponds to simple extension of the rod unaccompanied by flexure. In the third class of cases ϵ and κx are of the same order of magnitude. It is clear that if we retain quantities of the order ϵ and quantities of the order κx we shall be at liberty to reject at the end those terms which may be small in consequence of the case under discussion falling into the first or second class.

If with Poisson[1] and others we regard u, v, w as capable of expansion in powers of x, y, z with coefficients some functions of s, then it is clear that $\frac{\partial u}{\partial z}$, $\frac{\partial v}{\partial z}$, $\frac{\partial w}{\partial z}$ can also be expanded in the same form, but the terms that occur are of a lower order in x, y, z than the corresponding terms of u, v, w. The differential coefficients $\frac{\partial u}{\partial s}$, $\frac{\partial v}{\partial s}$, $\frac{\partial w}{\partial s}$ will be replaced by series of the same form as u, v, w, and the terms that occur will be of the same order. We may therefore for a first approximation neglect the terms $\frac{\partial u}{\partial s}$, $\frac{\partial v}{\partial s}$, $\frac{\partial w}{\partial s}$ and the terms such as $-\tau v + \lambda w$. Without however introducing the idea of series for u, v, w we can see at once that as u, v, w are small compared with x, y, z we ought for a first approximation to neglect the terms that are linear in u, v, w in comparison with the terms that are linear in x, y, z. With a little consideration we can also see that the differential coefficients such as $\frac{\partial u}{\partial s}$ are small compared with the differential coefficients such as $\frac{\partial u}{\partial z}$. For $\frac{\partial u}{\partial s}$ is the limit of a ratio in which the numerator is the difference of the values of u at homologous points of contiguous elementary prisms, and the denominator is the length of such a prism; while $\frac{\partial u}{\partial z}$ represents the rate of variation of u as we pass along a line parallel to the axis z from point to point of a single prism. Now u vanishes at the centre of each prism, and at homologous points of contiguous prisms the values of u are much more nearly equal than at the centre and an extreme point of a single prism[2].

[1] *Mém. Paris Acad.* VIII. 1829.

[2] This explanation was given by Saint-Venant in a footnote to the 'Annotated Clebsch,' p. 422. M. Boussinesq in his memoir of 1871 after obtaining Kirchhoff's identities declares that in his opinion this process of approximation is wanting in rigour. He admits that terms like κv may be neglected in comparison with terms like κy, but he says that there is nothing to shew that $\partial w/\partial s$ is not actually of the same order as ϵ, and he contends that in such a case ϵ and $\partial w/\partial s$ should both be retained. A reference to our classification of cases will remove this objection.

Both lines of argument thus lead to the same method of approximation, and we proceed to sketch the process. First leaving out in equations (7) the terms such as $\frac{\partial u}{\partial s}$, $-\tau v$, λw, and replacing $(1+\epsilon)$ by unity, we obtain equations which can be solved and which give a first approximation to the forms of u, v, w as functions of z. The solution will contain arbitrary functions of x and y. From the solutions obtained we may deduce approximations to the components of strain involving the arbitrary functions. The equations of equilibrium or small motion, and the boundary-conditions at the free surface of the prism will enable us to determine the arbitrary functions. Thus a first approximation to the values of u, v, w will be found. To obtain a second approximation[1] we may substitute the values found as a first approximation in the terms of equations (7) previously neglected and solve again. Recourse must again be had to the differential equations of equilibrium or small motion and the boundary-conditions in order to determine the arbitrary functions. It is evident that the process might be continued indefinitely, but would become very laborious. The first approximation may however be easily obtained and the work is very much simplified by using Kirchhoff's principle explained in art. 249, according to which it will be sufficient for this approximation to reduce the differential equations to a form independent of bodily forces and kinetic reactions.

253. First approximation. Internal Strain.

According to the method explained we reduce equations (7) to

$$\left.\begin{aligned} \frac{\partial u}{\partial z} &= -\tau y + \lambda z, \\ \frac{\partial v}{\partial z} &= -\kappa z + \tau x, \\ \frac{\partial w}{\partial z} &= -\lambda x + \kappa y + \epsilon \end{aligned}\right\} \quad \text{......................(8)},$$

If ϵ and $\partial w/\partial s$ are really of the same order of magnitude, then both are small in comparison with such terms as κy; and ϵ as well as $\partial w/\partial s$ ought to be rejected. No harm however is done by retaining ϵ temporarily and rejecting it at the end of the first approximation. The problem would be one of flexure and this would be the course which we should actually pursue.

[1] The second approximation is not required in our method, but the reader who wishes to see how it might be carried out is referred to the corresponding part of the theory of thin shells in ch. XXI.

by omitting terms containing such quantities as u, $\frac{\partial u}{\partial s}$, and replacing $1+\epsilon$ by unity. Integrating these equations we get

$$\left.\begin{aligned} u &= u_0 - \tau yz + \tfrac{1}{2}\lambda z^2, \\ v &= v_0 - \tfrac{1}{2}\kappa z^2 + \tau zx, \\ w &= w_0 - \lambda zx + \kappa zy + \epsilon z \end{aligned}\right\} \quad \text{..................(9)},$$

in which u_0, v_0, w_0 are functions of x and y.

From these we deduce the strain-components in the forms

$$\left.\begin{aligned} e &= \frac{\partial u_0}{\partial x}, & a &= \frac{\partial w_0}{\partial y} + \tau x, \\ f &= \frac{\partial v_0}{\partial y}, & b &= \frac{\partial w_0}{\partial x} - \tau y, \\ g &= \epsilon - \lambda x + \kappa y, & c &= \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \end{aligned}\right\} \text{.........(10)},$$

and we notice that, to the order of approximation to which the work has been carried, the strains, and therefore also the stresses, are independent of z. The order of approximation in question is such that the strains are of the first order in the small quantities x, y, z.

The stresses being independent of z the equations of equilibrium take the forms

$$\frac{\partial P}{\partial x} + \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial x} + \frac{\partial Q}{\partial y} = 0, \quad \frac{\partial T}{\partial x} + \frac{\partial S}{\partial y} = 0 \quad \text{...(11)},$$

in which bodily forces and kinetic reactions are omitted for the reasons explained in art. 249.

If l, m be the cosines of the angles which the normal to the cylindrical boundary makes with the axes of x and y, the conditions that hold at this boundary, when *no external surface-tractions are applied to it*, are

$$lP + mU = 0, \quad lU + mQ = 0, \quad lT + mS = 0 \text{......(12)}.$$

254. Torsional Rigidity.

In what follows we shall assume that the plane of (x, y) which is perpendicular to the elastic central-line of the rod is a plane of symmetry of the material. In this case the potential energy per unit of volume (W) can be written in the form (I. p. 81),

$$W = \tfrac{1}{2}(a_{11}, a_{22}, a_{33}, a_{66}, a_{12} \ldots \between e, f, g, c)^2 + \tfrac{1}{2}(a_{44}, a_{55}, a_{45} \between a, b)^2 \text{...(13)}.$$

Also the components of stress are the partial differential coefficients of W with respect to the components of strain, and thus in particular S and T are linear functions of a and b. The third of the equations of equilibrium (11) becomes

$$a_{55}\frac{\partial b}{\partial x}+a_{45}\left(\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}\right)+a_{44}\frac{\partial a}{\partial y}=0 \qquad (14),$$

or

$$a_{55}\frac{\partial^2 w_0}{\partial x^2}+a_{44}\frac{\partial^2 w_0}{\partial y^2}+2a_{45}\frac{\partial^2 w_0}{\partial x\partial y}=0 \qquad (15).$$

Also the third of the boundary-conditions (12) becomes

$$a_{55}lb+a_{45}(la+mb)+a_{44}ma=0 \qquad (16),$$

or

$$\left(a_{55}\frac{\partial w_0}{\partial x}+a_{45}\frac{\partial w_0}{\partial y}\right)\frac{dy}{ds_1}-\left(a_{44}\frac{\partial w_0}{\partial y}+a_{45}\frac{\partial w_0}{\partial x}\right)\frac{dx}{ds_1}$$
$$=\tau(a_{55}y-a_{45}x)\frac{dy}{ds_1}+(a_{44}x-a_{45}y)\frac{dx}{ds_1} \qquad (17),$$

where ds_1 is the element of arc of the bounding curve of the cross-section.

From this it appears that w_0 contains τ as a factor, and is independent of κ, λ, ϵ. In other words w_0 depends simply on the twist τ; and the strains a and b, and the stresses S and T are linear in τ. This result holds to the same order of approximation as equations (10).

It follows from what has just been said that there exists a quantity C such that

$$\iint(Sa+Tb)\,dxdy=C\tau^2 \qquad (18),$$

where the dimensions of C are those of the product of a modulus of elasticity and a moment of inertia of the cross-section of the rod. The quantity C is called the *torsional rigidity* of the rod. For the more symmetrical case there treated it is identical with the quantity introduced in I. art. 93.

255. Saint-Venant's Stress-conditions[1].

We may shew that the expressions (10) for the strain-components lead to Saint-Venant's stress-conditions that P, U, Q vanish.

For let ϕ and ψ be any two functions of x and y continuous

[1] See I. p. 148, equations (3).

within the bounding curve of the cross-section of the rod, and consider the integral

$$\iint\left\{P\frac{\partial\phi}{\partial x}+Q\frac{\partial\psi}{\partial y}+U\left(\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\right)\right\}dxdy,$$

the integration extending over the cross-section.

Using Green's transformation (I. p. 58) we can express this integral in the form

$$\int\{(lP+mU)\,\phi+(lU+mQ)\,\psi\}\,ds_1$$
$$-\iint\left\{\left(\frac{\partial P}{\partial x}+\frac{\partial U}{\partial y}\right)\phi+\left(\frac{\partial U}{\partial x}+\frac{\partial Q}{\partial y}\right)\psi\right\}dxdy.$$

The surface-integral vanishes identically by the first two of equations (11) and the line-integral by the first two of equations (12), so that the integral in question is equal to zero whatever continuous functions ϕ and ψ may be.

If in the integral we write successively $\psi=0$ and $\phi=x,\ y,\ x^2,\ xy,\ y^2$, and $\phi=0$ and $\psi=x,\ y,\ x^2,\ xy,\ y^2$, we find formulæ which may be written

$$\iint(P,\text{ or }Q,\text{ or }U)\,(1,\text{ or }x,\text{ or }y)\,dxdy=0\ldots\ldots\ldots(19).$$

Again, if we put $\phi=u$, $\psi=v$, we find

$$\iint(Pe+Qf+Uc)\,dxdy=0\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

Now the expression (13) for the potential energy W may be written in the form

$$W=\tfrac{1}{2}(A_{11},\ A_{22},\ A_{33},\ A_{66},\ A_{12}\ldots\ldots \between P,\ Q,\ R,\ U)^2$$
$$+\tfrac{1}{2}(A_{44},\ A_{55},\ A_{45}\between S,\ T)^2\ldots\ldots(21),$$

where the A's are reciprocal coefficients to the a's in the expression (13), and the quadratic function

$$(A_{11},\ A_{22},\ A_{33},\ A_{66},\ A_{12}\ldots\ldots \between P,\ Q,\ R,\ U)^2$$

is therefore essentially positive for all real values of P, Q, R, U.

Consider the function V defined by the equation

$$V=Pe+Qf+Uc-g\,(A_{13}P+A_{23}Q+A_{63}U)/A_{33},$$

in which
$$e=A_{11}P+A_{12}Q+A_{13}R+A_{16}U,$$
$$f=A_{21}P+A_{22}Q+A_{23}R+A_{26}U,$$
$$g=A_{31}P+A_{32}Q+A_{33}R+A_{36}U,$$
$$c=A_{61}P+A_{62}Q+A_{63}R+A_{66}U,$$

and $A_{rs}=A_{sr}$, $(r,\ s=1,\ 2,\ 3,\ 6)$.

We find at once that

$$A_{33}V = (A_{11}A_{33} - A_{13}^2)P^2 + (A_{22}A_{33} - A_{23}^2)Q^2 + (A_{66}A_{33} - A_{36}^2)U^2 + 2(A_{26}A_{33} - A_{23}A_{36})QU + 2(A_{16}A_{33} - A_{13}A_{36})UP + 2(A_{12}A_{33} - A_{13}A_{23})PQ,$$

and it can be readily verified that the conditions that V may be always positive are identical with the conditions that

$$(A_{11}, A_{22}, A_{33}, A_{66}, A_{12}, \ldots\ldots \between P, Q, R, U)^2$$

may be always positive[1].

Now $\iint dxdyg\,(A_{13}P + A_{23}Q + A_{63}U)/A_{33}$ vanishes identically by (19) since g is a linear function of x and y. Hence

$$\iint (Pe + Qf + Uc)\,dxdy$$

is equal to the integral over the cross-section of a quadratic function of P, Q, U which is always positive. It can therefore vanish only if $P = Q = U = 0$, and it has been proved to vanish in equation (20).

It thus appears that Saint-Venant's stress-conditions

$$P = Q = U = 0$$

are analytically deducible from the first two of the equations of equilibrium (11), the first two boundary-conditions (12), and the fact that the strain-component g is a linear function of the coordinates x, y of a point on the cross-section.

This proof is equivalent to that given by M. Boussinesq. In his work the first two of the equations of equilibrium in the forms given in (11) and the fact that g is a linear function of x and y are arrived at by considerations different from those here advanced[2].

The most general values of u, v, w which satisfy the equations of equilibrium[3] subject to Saint-Venant's stress-conditions have been given in I. ch. VII. If in these solutions all terms above the second order in x, y, z be omitted they will be found to be included in the forms given for u, v, w in (10).

[1] The verification involves a particular case of a theorem enunciated by Professor Sylvester and proved by Mr R. F. Scott, 'On Compound Determinants', *Proc. Lond. Math. Soc.* XIV. 1883.

[2] Liouville's *Journal*, XVI. 1871, pp. 148—151.

[3] A particular kind of symmetry of the material was assumed.

It is to be noticed that, although the expressions that we have obtained, and those that we shall obtain in the next article, have been found on the express understanding that no tractions are applied to the initially cylindrical bounding surface of the rod, yet they may still be used when such tractions are applied. In such a case the resultant for any elementary prism of the tractions in question will have to be taken among the forces directly applied to the rod, *i.e.* it must be included in the system (X, Y, Z, L, M, N) of art. 234.

256. Stress-system.

The general expression for the potential energy per unit volume is $\frac{1}{2}(Pe + Qf + Rg + Sa + Tb + Uc)$.

Since $P=0$, $Q=0$, and $U=0$, we have $R = Eg$, where E is the Young's modulus of the material for traction parallel to the elastic central-line of the rod. From this result and the remark at the end of art. 219 it follows that the potential energy of strain per unit length of the elastic central-line is

$$\iint \tfrac{1}{2} E g^2 dx dy + \tfrac{1}{2} C\tau^2.$$

The supposition that has been made as to the situation of the axes of x and y in the cross-section, viz. that they are the principal axes of the section at its centroid, shews that, on substituting for g, the potential energy per unit of length will be expressed by

$$\tfrac{1}{2}(EI_1\lambda^2 + EI_2\kappa^2 + C\tau^2 + E\omega\epsilon^2),$$

where ω is the area of the section, and, as in I. art. 92, $I_1 = \iint x^2 dx dy$ and $I_2 = \iint y^2 dx dy$. We thus have

$$W = \tfrac{1}{2}\int (EI_1\lambda^2 + EI_2\kappa^2 + C\tau^2 + E\omega\epsilon^2)\, ds \text{.........(22)},$$

the integral being taken along the elastic central-line of the rod. When the axes of x and y are chosen as above specified the axes of x, y, z are called the *principal torsion-flexure axes* of the rod.

Since κ, λ, τ, ϵ are independent quantities we could at once shew by variation of the energy that, to the order of approximation to which the work has been carried, there exists a resultant normal stress across any section equal to $\iint E\epsilon dx dy$ tending to increase ϵ, and three couples about the axes equal to $EI_2\kappa$, $EI_1\lambda$, and $C\tau$ tending to increase the component curvatures κ, λ and the twist τ. This result may be verified directly.

We have, for the resultant stresses across the section $z=0$ of the prism, forces

$$\iint T dx dy, \quad \iint S dx dy, \quad \iint R dx dy$$

parallel to the axes, and couples

$$\iint R y dx dy, \quad \iint - R x dx dy, \quad \iint (Sx - Ty) dx dy$$

about the axes. Now it is easy to shew that the first two forces vanish identically. Taking for example $\iint S dx dy$ its value, by (13), is equal to $\iint (a_{44} a + a_{45} b) dx dy$, and this may be written

$$\iint \left\{ a_{44} \frac{\partial}{\partial y}(ay) + a_{45} \left\{ \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial x}(ay) \right\} + a_{55} \frac{\partial}{\partial x}(by) \right\} dx dy$$

since the differential equation (14) holds at all points to which the integration extends.

The surface-integral just written can be transformed into the line-integral

$$\int \{ a_{44} m a + a_{45}(la + mb) + a_{55} lb \} y ds_1$$

taken round the bounding-curve of the normal section, and this vanishes identically by (16).

In like manner we may shew that $\iint T dx dy$ vanishes identically.

The first approximation therefore reduces the stress across any section to a tension parallel to the elastic central-line of the rod, and three couples.

The tension is

$$\iint R dx dy = \iint E(\epsilon - \lambda x + \kappa y) dx dy$$
$$= E\omega\epsilon,$$

since the origin is at the centroid of the section, so that this tension depends simply on the extension of the elastic central-line.

The couples are easily seen to be

$$EI_2\kappa, \quad EI_1\lambda, \quad C\tau,$$

where the notation has been already explained.

257. Application of the method of approximation.

In art. 252 we have explained a method of approximation whereby more exact values could be found for the displacements, strains, and stresses. The additional terms in the displacements would be of the third or a higher order in the coordinates x, y, z.

The strains and stresses to which they give rise would be of the second or higher order in the same quantities, and the corrections to the resultant stresses would consist of terms which could be of the same order of magnitude as the expressions already found for the stress-couples, while the corrections to the stress-couples would be of a higher order in the linear dimensions of the cross-section.

We must conclude that in the first and third classes of cases explained in art. 252 there exist two resultant shearing stresses N_1, N_2, and a tension T parallel to the axes of x, y, z, which are unknown without a second approximation, and couples G_1, G_2, H about the same axes of which sufficiently approximate values are given by the formulæ

$$G_1 = EI_2\kappa = A\kappa, \quad G_2 = EI_1\lambda = B\lambda, \quad H = C\tau \quad \text{......(23)}.$$

In the second class of cases there exists a tension T which is given by the equation

$$T = E\omega\epsilon \quad \text{.............................(24)},$$

and the remaining stress-resultants and stress-couples are unimportant.

The quantities A and B are called as in art. 213 the *principal flexural rigidities* of the rod, and the quantity C is the *torsional rigidity*.

The application of the method of approximation would also give rise to a correction to the expression (22) for the potential energy, but the correction is unimportant. In using this expression we must omit the term in ϵ in the first class of cases, and the terms in κ, λ, τ in the second class of cases, while in the third class of cases both terms must be retained. With the notation (23) the expression for the potential energy per unit length of the rod is

$$\tfrac{1}{2}\left[A\kappa^2 + B\lambda^2 + C\tau^2\right] + \tfrac{1}{2}E\omega\epsilon^2 \quad \text{...............(25)},$$

of which generally only the first term or only the second need be retained.

In the theory of the equilibrium of the rod under given forces each particular case will fall into one or other of the three classes. The third class is very special and will only be met with in exceptional conditions. Excluding it we may say that either

the deformation is characterised by flexure or torsion unaccompanied by extension, or else the extension is the governing circumstance. For example, if a straight prismatic wire or rod whose axis is vertical be supported at its upper extremity and loaded at its lower extremity it will be simply extended. If the same wire or rod have one extremity built in so as to make a finite angle with the vertical and be loaded at the other extremity it will be bent without stretching; but if the angle it makes with the vertical be continually diminished we may come upon a series of positions forming a transition from conditions of simple flexure to conditions of simple extension, and these will fall under the third class.

258. Kinetic Energy.

We shall hereafter require an expression for the kinetic energy when the rod is in motion. For this we calculate the velocity of any point on a normal section and integrate over the normal section, this gives the kinetic energy per unit length of the rod.

Let P be any point on the central-line whose coordinates after strain referred to fixed axes are ξ, η, ζ, and let Q be the point whose coordinates before strain referred to the three line-elements at P are x, y, z. Then, by (2) of art. 232, the coordinates of Q after strain referred to the axes of ξ, η, ζ are

$$\xi + l_1(x+u) + l_2(y+v) + l_3(z+w),$$
$$\eta + m_1(x+u) + m_2(y+v) + m_3(z+w),$$
$$\zeta + n_1(x+u) + n_2(y+v) + n_3(z+w),$$

and if Q be in the normal section through P we may omit z. In these expressions $\xi, \eta, \zeta, u, v, w$, and the l's, m's, and n's are functions of the time.

Now putting $z=0$ and omitting u, v, w as small compared with x, y we obtain an approximation to the component velocities in the form

$$\frac{\partial \xi}{\partial t} + \frac{\partial l_1}{\partial t}x + \frac{\partial l_2}{\partial t}y,$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial m_1}{\partial t}x + \frac{\partial m_2}{\partial t}y,$$
$$\frac{\partial \zeta}{\partial t} + \frac{\partial n_1}{\partial t}x + \frac{\partial n_2}{\partial t}y,$$

so that the kinetic energy per unit length is

$$\tfrac{1}{2}\rho\omega\left\{\left(\frac{\partial\xi}{\partial t}\right)^2+\left(\frac{\partial\eta}{\partial t}\right)^2+\left(\frac{\partial\zeta}{\partial t}\right)^2\right\}+\tfrac{1}{2}\rho I_1\left\{\left(\frac{\partial l_1}{\partial t}\right)^2+\left(\frac{\partial m_1}{\partial t}\right)^2+\left(\frac{\partial n_1}{\partial t}\right)^2\right\}$$
$$+\tfrac{1}{2}\rho I_2\left\{\left(\frac{\partial l_2}{\partial t}\right)^2+\left(\frac{\partial m_2}{\partial t}\right)^2+\left(\frac{\partial n_2}{\partial t}\right)^2\right\}\quad\text{......(26)},$$

where ρ is the density of the material.

CHAPTER XVI.

THEORY OF THE SMALL VIBRATIONS OF THIN RODS.

259. WE propose in this chapter to deduce the equations of vibration of thin rods from Kirchhoff's theory explained in chapter XV., and afterwards to sketch the process by which those modes of vibration of an isotropic circular cylinder can be obtained which correspond with the modes that we shall find for a thin rod. It is unnecessary here to enter into a minute discussion of the modes in question, as they are fully considered in Lord Rayleigh's *Theory of Sound*, vol. I. chapters VII. and VIII. For details the reader is referred to that work.

260. The Variational Equation[1].

According to I. art. 64, the general variational equation of small vibration of a solid under forces applied at the boundary alone can be written

$$\iiint -\rho\left(\frac{\partial^2 u}{\partial t^2}\delta u+\frac{\partial^2 v}{\partial t^2}\delta v+\frac{\partial^2 w}{\partial t^2}\delta w\right)dx\,dy\,dz-\iiint \delta W\,dx\,dy\,dz$$
$$+\iint (F\delta u+G\delta v+H\delta w)\,dS=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where W is the potential energy of the strained solid per unit volume, ρ is the density, and u, v, w are displacements parallel to fixed axes. In the case of thin elastic bodies such as bars or plates it is convenient to give a special form to the terms like $\iiint -\rho\frac{\partial^2 u}{\partial t^2}\delta u\,dx\,dy\,dz$ that arise from kinetic reactions, viz. we notice that these terms in the left-hand member of (1) are

[1] Kirchhoff, *Vorlesungen über Mathematische Physik, Mechanik.*

the halves of those that would occur if we formed the variation of the "action" A, where

$$A = \int dt \iiint \rho \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right\} dx dy dz.$$

For this gives us

$$\tfrac{1}{2}\delta A = \left[\rho \left(\frac{\partial u}{\partial t}\,\delta u + \frac{\partial v}{\partial t}\,\delta v + \frac{\partial w}{\partial t}\,\delta w\right)\right]$$
$$- \int dt \iiint \rho \left(\frac{\partial^2 u}{\partial t^2}\,\delta u + \frac{\partial^2 v}{\partial t^2}\,\delta v + \frac{\partial^2 w}{\partial t^2}\,\delta w\right) dx dy dz,$$

where the part in square brackets is to be taken between limits.

It follows that the equations of vibration can be found by equating to zero the variation

$$\delta \int dt \iiint (T - W)\, dx dy dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where T is the kinetic energy per unit volume, and W is the potential energy of strain. This is really the application of the principle of Least Action in Elasticity.

261. Kinematics of small deformation[1].

Let u, v, w be the displacements of a point P on the elastic central-line of a thin rod parallel to fixed axes of ξ, η, ζ, of which the axis ζ is parallel to the unstrained elastic central-line, and the axis ξ to the transverse (1), and let $\mathrm{u} + d\mathrm{u}$, $\mathrm{v} + d\mathrm{v}$, $\mathrm{w} + d\mathrm{w}$ be the displacements of a neighbouring point P' on the elastic central-line. Let ξ, η, ζ be the coordinates of P after strain, and $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$ those of P', then

$$d\xi = d\mathrm{u}, \quad d\eta = d\mathrm{v}, \quad d\zeta = ds + d\mathrm{w}.$$

If ds' be the stretched length of PP' we have

$$(ds')^2 = (ds)^2 \left[\left(\frac{d\mathrm{u}}{ds}\right)^2 + \left(\frac{d\mathrm{v}}{ds}\right)^2 + \left(1 + \frac{d\mathrm{w}}{ds}\right)^2\right].$$

If ϵ be the extension so that $ds' = ds\,(1 + \epsilon)$, we shall have, rejecting terms of the second order in u, v, w,

$$\epsilon = d\mathrm{w}/ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Let l_3, m_3, n_3 be the direction-cosines of the tangent to the elastic central-line after strain, referred to the axes of ξ, η, ζ, then

$$l_3 = d\xi/ds', \quad m_3 = d\eta/ds', \quad n_3 = d\zeta/ds'.$$

[1] See arts. 232 sq. for the notation.

Rejecting terms of the second order in u, v, w, these become

$$l_3 = d\mathrm{u}/ds, \quad m_3 = d\mathrm{v}/ds, \quad n_3 = 1 \;\ldots\ldots\ldots\ldots\ldots(4).$$

After strain the line-element (1) which was initially parallel to the axis ξ will make an angle $\frac{1}{2}\pi - \gamma$ with the axis ζ, where γ is small, and the plane through the line-elements (1) and (3) which was initially parallel to the plane (ξ, ζ) will make a very small angle β with that plane. The direction-cosines of the strained position of the line-element (1) are $1, \beta, \gamma$, rejecting terms of the second order in the small quantities. To find the direction-cosines (l_1, m_1, n_1) of the axis x, which is in the plane through the lines (l_3, m_3, n_3) and $(1, \beta, \gamma)$ and perpendicular to (l_3, m_3, n_3), we have the equations

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ 1, & \beta, & \gamma \\ l_3, & m_3, & n_3 \end{vmatrix} = 0,$$

and $$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0.$$

Since l_3 and m_3 as well as β and γ are small, we find ultimately

$$l_1 = 1, \quad m_1 = \beta, \quad n_1 = -l_3 = -d\mathrm{u}/ds \;\ldots\ldots\ldots\ldots(5).$$

In like manner the direction-cosines l_2, m_2, n_2 of the axis y are found to be

$$l_2 = -\beta, \quad m_2 = 1, \quad n_2 = -m_3 = -d\mathrm{v}/ds \;\ldots\ldots\ldots\ldots(6).$$

We can now calculate the quantities κ, λ, τ introduced and defined in art. 232. We defined κ and λ to be the component curvatures of the elastic central-line about the axes of x and y, and τ is the twist. We have, by equations (4) of art. 232,

$$\kappa ds' = l_3 dl_2 + m_3 dm_2 + n_3 dn_2 = -(l_2 dl_3 + m_2 dm_3 + n_2 dn_3)$$
$$= -\left[-\beta d\left(\frac{d\mathrm{u}}{ds}\right) + d\,\frac{d\mathrm{v}}{ds}\right];$$

so that, rejecting small quantities of the second order in β, u, v, w, we have

$$\kappa = -\frac{d^2\mathrm{v}}{ds^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Again, $$\lambda ds' = l_1 dl_3 + m_1 dm_3 + n_1 dn_3$$
$$= d\left(\frac{d\mathrm{u}}{ds}\right) + \beta d\left(\frac{d\mathrm{v}}{ds}\right);$$

and, rejecting small quantities of the second order, we have

$$\lambda = \frac{d^2 \mathrm{u}}{ds^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Lastly, we have

$$\begin{aligned} \tau ds' &= l_2 dl_1 + m_2 dm_1 + n_2 dn_1 \\ &= d\beta + \frac{d\mathrm{v}}{ds} d\left(\frac{d\mathrm{u}}{ds}\right); \end{aligned}$$

and, rejecting small quantities of the second order, we have

$$\tau = \frac{d\beta}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

262. First method of forming the equations of Vibration.

According to art. 257 the potential energy of the extended bent and twisted rod is

$$\tfrac{1}{2}\int\left\{A\left(\frac{d^2\mathrm{v}}{ds^2}\right)^2 + B\left(\frac{d^2\mathrm{u}}{ds^2}\right)^2 + C\left(\frac{d\beta}{ds}\right)^2 + E\omega\left(\frac{d\mathrm{w}}{ds}\right)^2\right\} ds \ldots (10),$$

where A and B are the principal flexural rigidities, C the torsional rigidity, E the Young's modulus for pull in the direction of the elastic central-line, and ω the area of the normal section in the unstrained state.

By the result of art. 258 we have for the kinetic energy

$$\tfrac{1}{2}\int\rho\left[\omega(\dot{\mathrm{u}}^2 + \dot{\mathrm{v}}^2 + \dot{\mathrm{w}}^2) + I_1\left\{\dot{\beta}^2 + \left(\frac{\partial^2 \mathrm{u}}{\partial s \partial t}\right)^2\right\} + I_2\left\{\dot{\beta}^2 + \left(\frac{\partial^2 \mathrm{v}}{\partial s \partial t}\right)^2\right\}\right] ds$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

where ρ is the density of the material, and I_1, I_2 are the moments of inertia of the normal section about its principal axes, and the dots denote differentiation with respect to the time.

If we write

$$I_1 = \omega k_1^2, \quad I_2 = \omega k_2^2, \quad I_1 + I_2 = \omega k^2 \ldots\ldots\ldots\ldots (12),$$

so that the k's are the radii of gyration of the normal section about principal axes at its centroid, we have for the potential energy of strain

$$W_1 = \tfrac{1}{2}\int\left[C\left(\frac{\partial\beta}{\partial s}\right)^2 + E\omega\left\{k_2^2\left(\frac{\partial^2 \mathrm{v}}{\partial s^2}\right)^2 + k_1^2\left(\frac{\partial^2 \mathrm{u}}{\partial s^2}\right)^2 + \left(\frac{\partial \mathrm{w}}{\partial s}\right)^2\right\}\right] ds \ldots (13),$$

and for the kinetic energy

$$T_1 = \tfrac{1}{2}\int \rho\omega \left[\left(\frac{\partial u}{\partial t}\right)^2 + k_1^2\left(\frac{\partial^2 u}{\partial s \partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 + k_2^2\left(\frac{\partial^2 v}{\partial s\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 + k^2\left(\frac{\partial \beta}{\partial t}\right)^2\right] ds \;\ldots\ldots\ldots\ldots (14).$$

To form now the equations of vibration we have to form the variation

$$\delta\int(T_1 - W_1)\,dt$$

and equate separately to zero the terms in δu, δv, δw, and $\delta\beta$. It is clear that the terms in δu contain u but no v or w or β, and similarly for the others, and therefore the vibrations fall into three classes:—

(1) *Extensional vibrations* involving only w.

(2) *Torsional vibrations* involving only β.

(3) *Flexural vibrations* involving only u, or v. The equations for vibrations involving u are of the same form as those for vibrations involving v, the coefficients only being different.

263. Extensional Vibrations.

Taking first the extensional vibrations we have to equate to zero the variation

$$\delta\iint\tfrac{1}{2}\left\{\rho\omega\left(\frac{\partial w}{\partial t}\right)^2 - E\omega\left(\frac{\partial w}{\partial s}\right)^2\right\} ds\,dt.$$

We thus obtain

$$\iint\left(\rho\omega\frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t} - E\omega\frac{\partial w}{\partial s}\frac{\partial \delta w}{\partial s}\right) ds\,dt = 0,$$

or

$$\int\rho\omega\left[\frac{\partial w}{\partial t}\,\delta w\right] ds - \int E\omega\left[\frac{\partial w}{\partial s}\,\delta w\right] dt - \iint\left\{\rho\omega\frac{\partial^2 w}{\partial t^2} - E\omega\frac{\partial^2 w}{\partial s^2}\right\}\delta w\,ds\,dt = 0.$$

From the vanishing of the terms under the double sign of integration we obtain the equation of vibration

$$\frac{\partial^2 w}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 w}{\partial s^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

The vibrations here considered are the "longitudinal" vibrations of Lord Rayleigh's *Theory of Sound*. We have described

them as "extensional" to avoid the suggestion that there is no lateral motion of the parts of the rod. The only resultant stress existing in the rod is the tension across the normal sections, and the extension in the direction of the elastic central-line is accompanied by lateral contraction. The terms of the kinetic energy arising from the inertia of the lateral motion are of the order neglected, the terms of the potential energy arising from the lateral contraction are of the same order as those arising from the longitudinal extension. If the lateral motion were altogether neglected the coefficient of $\partial^2 \mathrm{w}/\partial s^2$ in (15) would not be E/ρ. For example, if the material were isotropic it would be $(\lambda + 2\mu)/\rho$.

The functional solution of the differential equation (15) is

$$\mathrm{w} = f(s - at) + F(s + at) \qquad \text{.................. (16)},$$

where $a^2 = E/\rho$, and it represents waves of extension or contraction travelling along the rod with velocity $\sqrt{(E/\rho)}$.

The form of the normal functions depends on the terminal conditions. At a fixed end we have $\mathrm{w} = 0$, and at a free end the tension $E\partial \mathrm{w}/\partial s$ vanishes. For a rod of length l free at both ends the normal functions are of the form $\cos n\pi s/l$, where n is an integer and s is measured from one end. The solution of the differential equation in terms of normal functions and normal coordinates is

$$\mathrm{w} = \sum_n \cos n\pi s/l \, \{A_n \cos n\pi at/l + B_n \sin n\pi at/l\}.$$

264. Torsional Vibrations.

For the torsional vibrations we have to equate to zero the variation

$$\delta \iint \tfrac{1}{2} \left\{ \rho\omega k^2 \left(\frac{\partial \beta}{\partial t}\right)^2 - C \left(\frac{\partial \beta}{\partial s}\right)^2 \right\} ds\,dt.$$

Proceeding as in the last case we find the differential equation of vibration

$$\rho\omega k^2 \frac{\partial^2 \beta}{\partial t^2} = C \frac{\partial^2 \beta}{\partial s^2} \qquad \text{........................ (17)}.$$

In this equation $\rho\omega$ is the mass of a unit of length of the rod, k the radius of gyration of the normal section about the elastic central-line, and C the torsional rigidity.

The solutions of the equation are of the same form as those of the equation for extensional vibrations, but the velocity of propagation of torsional waves is $\sqrt{(C/\rho\omega k^2)}$.

Referring to the solution of the torsion-problem in I. ch. VI. we see that, in case the material has three planes of symmetry of which one coincides with the normal section of the rod, and the other two are the planes through its elastic central-line (3) and the principal axes of inertia of its normal sections (1) and (2), and, in case the modulus of rigidity for the two directions (1, 3) is identical with that for the two directions (2, 3), we have approximately

$$C = \tfrac{1}{40}\frac{\omega^3}{k^2}\mu,$$

where μ is the modulus of rigidity in question. Thus the velocity of propagation of torsional waves is approximately

$$\frac{\omega}{k^2}\sqrt{\frac{\mu}{40\rho}}.$$

For a circular section of radius a

$$C = \tfrac{1}{2}\mu\pi a^4,$$

and the velocity is $\sqrt{(\mu/\rho)}$.

For an elliptic section of semiaxes a, b

$$C = \mu\pi a^3b^3/(a^2+b^2),$$

and the velocity is $\dfrac{2ab}{a^2+b^2}\sqrt{(\mu/\rho)}$.

If the rod have a free end, then the torsional couple $C\partial\beta/\partial s$ vanishes at that end. At a fixed end where the section is prevented from rotating about the elastic central-line β vanishes. The normal functions in the various cases can be easily investigated and are of the same form as in the corresponding cases of extensional vibrations.

265. Flexural Vibrations.

For the flexural vibrations in which u alone varies we have to equate to zero the variation

$$\delta\iint\tfrac{1}{2}\left[\rho\omega\left\{\left(\frac{\partial u}{\partial t}\right)^2 + k_1^2\left(\frac{\partial^2 u}{\partial s\partial t}\right)^2\right\} - E\omega k_1^2\left(\frac{\partial^2 u}{\partial s^2}\right)^2\right]ds\,dt \ldots(18).$$

In forming the variations we make use of the identities

$$\frac{\partial^2 u}{\partial s^2}\frac{\partial^2 \delta u}{\partial s^2} - \frac{\partial^4 u}{\partial s^4}\delta u = \frac{\partial}{\partial s}\left(\frac{\partial^2 u}{\partial s^2}\frac{\partial \delta u}{\partial s} - \frac{\partial^3 u}{\partial s^3}\delta u\right),$$

$$2\left(\frac{\partial^2 u}{\partial s\partial t}\frac{\partial^2 \delta u}{\partial s\partial t} - \frac{\partial^4 u}{\partial s^2\partial t^2}\delta u\right) = \frac{\partial}{\partial s}\left(\frac{\partial^2 u}{\partial s\partial t}\frac{\partial \delta u}{\partial t} - \frac{\partial^3 u}{\partial s\partial t^2}\delta u\right) + \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial s\partial t}\frac{\partial \delta u}{\partial s} - \frac{\partial^3 u}{\partial s^2\partial t}\delta u\right).$$

Integrating (18) by parts, and equating to zero the coefficient of δu under the sign of double integration, we obtain the differential equation

$$\rho\left\{\frac{\partial^2 u}{\partial t^2} - k_1^2\frac{\partial^4 u}{\partial s^2\partial t^2}\right\} + Ek_1^2\frac{\partial^4 u}{\partial s^4} = 0 \quad\text{.............(19).}$$

In this equation the first term arises from the inertia of the lateral motion, the second term from the "rotatory inertia", and the third term from the elastic resistance to flexure.

The term arising from rotatory inertia is in general negligible in comparison with that arising from the inertia of the lateral motion, and the differential equation may be reduced to

$$\frac{\partial^2 u}{\partial t^2} = -\frac{E}{\rho}k_1^2\frac{\partial^4 u}{\partial s^4} \quad\text{........................(20).}$$

This is the equation for flexural vibrations in the plane through the line-elements (1) and (3).

In like manner the equation for flexural vibrations in the plane through the line-elements (2) and (3) is

$$\frac{\partial^2 v}{\partial t^2} = -\frac{E}{\rho}k_2^2\frac{\partial^4 v}{\partial s^4} \quad\text{........................(21).}$$

To find the normal functions and the period-equation we shall have to introduce the terminal conditions. These are

(1) for a built-in end, $u = 0$ and $\partial u/\partial s = 0$ at the end;

(2) for a free end, $\partial^2 u/\partial s^2 = 0$ and $\partial^3 u/\partial s^3 = 0$ at the end;

(3) for an end simply supported, $u = 0$ and $\partial^2 u/\partial s^2 = 0$ at the end.

Let $p/2\pi$ be the frequency so that the equation (20) becomes

$$\frac{E}{\rho}k_1^2\frac{\partial^4 u}{\partial s^4} = p^2 u,$$

and let $$Ek_1^2/\rho p^2 = l^4/m^4 \quad\text{........................(22),}$$

where l is the length of the rod. The normal functions are of the form

$$A\cosh\frac{ms}{l}+B\sinh\frac{ms}{l}+C\cos\frac{ms}{l}+D\sin\frac{ms}{l}\quad\text{......(23)}.$$

The terminal conditions give four linear equations connecting the constants A, B, C, D, and the elimination of these leads to an equation for m. When m is found from this equation the frequency $p/2\pi$ is given by equation (22).

266. Particular cases.

If now we write $\qquad \mathrm{u}=\Sigma A_r \mathrm{u}_r e^{\iota p_r t}$........................(24),

u_r is a function of s of the form

$$A\cosh\frac{ms}{l}+B\sinh\frac{ms}{l}+C\cos\frac{ms}{l}+D\sin\frac{ms}{l},$$

where m is a root of a certain transcendental equation, and the ratios of $A:B:C:D$ are determined by the terminal conditions.

It is convenient to write down the equations for m and the forms of the normal functions in some particular cases. The reader may supply the proofs, or they will be found in Lord Rayleigh's *Theory of Sound*, vol. I. ch. VIII.

(α) For a rod free at both ends m is given by the equation

$$\cos m\cosh m=1\quad\text{........................(25)},$$

and for any value of m the corresponding value of u_r is

$$(\sin m-\sinh m)\left(\cos\frac{ms}{l}+\cosh\frac{ms}{l}\right)$$
$$-(\cos m-\cosh m)\left(\sin\frac{ms}{l}+\sinh\frac{ms}{l}\right)\quad\text{......... (26)}.$$

(β) For a rod built-in at both ends m is given by the equation (25), viz.:

$$\cos m\cosh m=1,$$

and for any value of m the corresponding value of u_r is

$$(\sin m-\sinh m)\left(\cos\frac{ms}{l}-\cosh\frac{ms}{l}\right)$$
$$-(\cos m-\cosh m)\left(\sin\frac{ms}{l}-\sinh\frac{ms}{l}\right)\quad\text{......... (27)}.$$

(γ) For a rod built-in at $s=0$, and free at $s=l$, the equation for m becomes

$$\cos m \cosh m = -1 \qquad\qquad (28),$$

and for any value of m the corresponding value of u_r is

$$(\sin m + \sinh m)\left(\cos\frac{ms}{l} - \cosh\frac{ms}{l}\right)$$

$$-(\cos m + \cosh m)\left(\sin\frac{ms}{l} - \sinh\frac{ms}{l}\right) \qquad\qquad (29).$$

(δ) For a rod freely supported at both ends we have

$$m = n\pi \qquad\qquad (30),$$

where n is an integer, and the normal functions are of the form

$$\sin\frac{n\pi s}{l} \qquad\qquad (31).$$

267. Second method of forming the Equations of Vibration.

The equations of vibration may also be deduced from (8) and (10) of art. 234, if we replace the forces and couples X, Y, Z and L, M, N by those arising from kinetic reactions.

(1) For extensional vibrations there will be no couples to take the place of L, M, N, and, since κ, λ, τ all vanish, G_1, G_2, H, and therefore also N_1 and N_2, all vanish. There will be no forces to take the place of X, Y and the equations reduce to

$$dT + Zds = 0,$$

where Zds is equal and opposite to the rate of change of momentum of the element between two normal sections distant ds.

Now in the notation of the present chapter this rate of change of momentum is $\rho\omega ds\dfrac{\partial^2\mathrm{w}}{\partial t^2}$, and T is $E\omega\dfrac{\partial\mathrm{w}}{\partial s}$. Hence the equation (15) of art. 263, viz.

$$\frac{\partial^2\mathrm{w}}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2\mathrm{w}}{\partial s^2}.$$

(2) For torsional vibrations κ, λ, and ϵ vanish. There are no couples L, M arising from kinetic reactions, and Nds is equal and opposite to the rate of change of the moment of momentum about

the axis z of the element between two normal sections distant ds, and this is $\rho\omega k^2 ds\partial^2\beta/\partial t^2$. The curvatures κ, λ and the couples G_1, G_2 vanish, while $H = Cd\beta/ds$; so that N_1 and N_2 vanish, and the third of equations (10) of art. 234 becomes equation (17) of art. 264, viz.

$$C\frac{d^2\beta}{ds^2} = \rho\omega k^2\frac{\partial^2\beta}{\partial t^2}.$$

(3) For the flexural vibrations we may suppose κ, τ and ϵ to vanish. Then H and G_1 vanish, and there are no couples L, N. The couple Mds is equal and opposite to the rate of change of the moment of momentum about the axis y of the element between two normal sections distant ds, and this is $\rho\omega k_1^2 ds\frac{\partial^2}{\partial t^2}\left(\frac{\partial \mathrm{u}}{\partial s}\right)$. The couple G_2 is $E\omega k_1^2\partial^2\mathrm{u}/\partial s^2$, and thus the second of equations (10) of art. 234 becomes

$$E\omega k_1^2\frac{\partial^3\mathrm{u}}{\partial s^3} + N_1 = \rho\omega k_1^2\frac{\partial^3\mathrm{u}}{\partial t^2\partial s} \quad\ldots\ldots\ldots\ldots\ldots (32).$$

Observing that such products as $N_1\lambda$ are of the second order in u, we see that the equations (8) of art. 234 can be reduced to

$$\frac{dN_1}{ds} = \rho\omega\frac{\partial^2\mathrm{u}}{\partial t^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

by substituting for $-Xds$, the quantity $\rho\omega ds\partial^2\mathrm{u}/\partial t^2$, which is the rate of change of the momentum parallel to x of the element between two normal sections distant ds.

Eliminating N_1 from the equations (32) and (33) we obtain

$$-\frac{Ek_1^2}{\rho}\frac{\partial^4\mathrm{u}}{\partial s^4} = \frac{\partial^2\mathrm{u}}{\partial t^2} - k_1^2\frac{\partial^4\mathrm{u}}{\partial s^2\partial t^2},$$

which is the same as equation (19) of art. 265.

Equation (20) can be deduced by neglecting the rotatory inertia or rejecting the right-hand member of equation (32), and in like manner equation (21) may be obtained by supposing λ, τ and ϵ to vanish.

It is worth while to remark that, if rotatory inertia be rejected, we have for the resultant shearing-stress N_1 at any section

$$N_1 = -\frac{d}{ds}\left[E\omega k_1^2\frac{\partial^2\mathrm{u}}{\partial s^2}\right] = -\frac{d}{ds}\,(\text{flexural couple})\ldots. (34).$$

This equation holds equally for equilibrium and for small motions.

268. Vibrations of a circular cylinder[1].

The theory just given, being derived from Kirchhoff's theory of thin rods, is confessedly a first approximation; and it is interesting to see, in a particular case, how the same results may be deduced from the general equations of small vibration of elastic solids. For this purpose we shall consider the theory of the free vibrations of an isotropic circular cylinder, whose radius will not at first be supposed small.

Referring to cylindrical coordinates r, θ, z the differential equations of small vibration in terms of displacements u, v, w along the radius, the circular section, and the generator through any point are, (I. p. 217,)

$$\left.\begin{aligned} \rho\frac{\partial^2 u}{\partial t^2} &= (\lambda+2\mu)\frac{\partial\Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial\varpi_3}{\partial\theta} + 2\mu\frac{\partial\varpi_2}{\partial z}, \\ \rho\frac{\partial^2 v}{\partial t^2} &= (\lambda+2\mu)\frac{1}{r}\frac{\partial\Delta}{\partial\theta} - 2\mu\frac{\partial\varpi_1}{\partial z} + 2\mu\frac{\partial\varpi_3}{\partial r}, \\ \rho\frac{\partial^2 w}{\partial t^2} &= (\lambda+2\mu)\frac{\partial\Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_2) + \frac{2\mu}{r}\frac{\partial\varpi_1}{\partial\theta} \end{aligned}\right\} \ldots(35),$$

in which

$$\left.\begin{aligned} \Delta &= \frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{\partial w}{\partial z}, \\ 2\varpi_1 &= \frac{1}{r}\left(\frac{\partial w}{\partial\theta} - \frac{\partial(rv)}{\partial z}\right), \\ 2\varpi_2 &= \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}, \\ 2\varpi_3 &= \frac{1}{r}\left(\frac{\partial(rv)}{\partial r} - \frac{\partial u}{\partial\theta}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots (36),$$

and ϖ_1, ϖ_2, ϖ_3 satisfy the identical relation

$$\frac{\partial(r\varpi_1)}{\partial r} + \frac{\partial\varpi_2}{\partial\theta} + \frac{\partial(r\varpi_3)}{\partial z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots(37).$$

[1] Pochhammer, 'Ueber die Fortpflanzungsgeschwindigkeit kleiner Schwingungen in einem unbegrenzten isotropen Kreiscylinder'. Crelle-Borchardt, LXXXI. 1876.

The stresses across a cylindrical surface $r = \text{const.}$ are

$$\widehat{rr} = \lambda\Delta + 2\mu\frac{\partial u}{\partial r},\ \widehat{\theta r} = \mu\left\{\frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right\},\ \widehat{zr} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\ldots(38),$$

and at the cylindrical boundary we shall suppose these to vanish.

The conditions at the ends of the cylinder will not be supposed given, but we shall investigate certain modes of vibration and the conditions that must hold in order that these vibrations may be executed.

The modes of vibration in question are those in which (as functions of z, θ and t) u, v and w are proportional to an exponential $e^{\iota(\beta\theta+\gamma z+pt)}$, where β is zero or an integer, p a real constant, and γ a real or imaginary constant. We write

$$u = Ue^{\iota(\beta\theta+\gamma z+pt)},\ v = Ve^{\iota(\beta\theta+\gamma z+pt)},\ w = We^{\iota(\beta\theta+\gamma z+pt)}\ldots(39),$$

and classify the vibrations according to the values assigned to β.

269. Torsional Vibrations[1].

Supposing first that $\beta = 0$, or that u, v, w are independent of θ, the second of the differential equations (35) reduces to

$$-\rho p^2 v = \mu\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial(rv)}{\partial r}\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{r}\frac{\partial(rv)}{\partial z}\right\}\right]$$

or

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} + V\left\{\frac{\rho p^2}{\mu} - \gamma^2 - \frac{1}{r^2}\right\} = 0.$$

The solution of this for space containing the axis $(r = 0)$ is

$$V = AJ_1(\kappa r),$$

where

$$\kappa^2 = \rho p^2/\mu - \gamma^2 \ldots\ldots\ldots\ldots\ldots\ldots(40),$$

J_1 denotes Bessel's function of order unity, and A is an arbitrary constant.

Now it is clear that all the differential equations of vibration will be satisfied by the solution

$$u = 0,\ w = 0,\ v = AJ_1(\kappa r)\,e^{\iota(\gamma z+pt)},$$

where the constants κ, γ, p are connected by equation (40).

[1] Cf. I. p. 147, footnote.

If c be the radius of the cylinder the conditions at the cylindrical boundary reduce to

$$\frac{d}{dc}\left\{\frac{J_1(\kappa c)}{c}\right\}=0,$$

which is satisfied by $\kappa=0$, or

$$\gamma=\pm p\sqrt{(\rho/\mu)} \quad\quad (41).$$

Since $\kappa=0$, we must reduce the Bessel's function to its first term r.

This solution gives us the torsional vibrations of a circular cylinder, and the velocity of propagation of torsional waves is $\sqrt{(\mu/\rho)}$.

The stress across a normal section $z=$ const. at any point has components $\widehat{zz}=0$, $\widehat{zr}=0$, and

$$\widehat{z\theta}=\mu\frac{\partial v}{\partial z}=\mu\iota\gamma Are^{\iota(\gamma z+pt)}.$$

The ends can be free if $A\iota e^{\iota\gamma z}$ vanish at both ends. Thus for a free-free rod of length l we have the solution given by the ordinary theory, viz. we have

$$\frac{v}{r}=\sum_n \cos\frac{n\pi z}{l}\left(A_n\cos\frac{n\pi t}{l}\sqrt{\frac{\mu}{\rho}}+B_n\sin\frac{n\pi t}{l}\sqrt{\frac{\mu}{\rho}}\right),$$

where n is an integer, and z is measured from one end.

270. Extensional Vibrations.

In the next place suppose v vanishes, but u and w do not vanish, β being zero as before. The second of the differential equations (35) is satisfied identically, and the first two become

$$\rho\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)\frac{\partial\Delta}{\partial r}+2\mu\frac{\partial\varpi_2}{\partial z},$$

$$\rho\frac{\partial^2 w}{\partial t^2}=(\lambda+2\mu)\frac{\partial\Delta}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\varpi_2).$$

From these by differentiation we form the equations

$$\left.\begin{aligned}\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{\partial^2\Delta}{\partial z^2}&=\frac{\rho}{\lambda+2\mu}\frac{\partial^2\Delta}{\partial t^2},\\ \frac{\partial^2\varpi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi_2}{\partial r}-\frac{\varpi_2}{r^2}+\frac{\partial^2\varpi_2}{\partial z^2}&=\frac{\rho}{\mu}\frac{\partial^2\varpi_2}{\partial t^2}\end{aligned}\right\} \quad\quad (42);$$

and, by substituting $-\gamma^2$ for $\partial^2/\partial z^2$, and $-p^2$ for $\partial^2/\partial t^2$, we find that ϖ_2 is proportional to $J_1(\kappa r)$, where κ is given by (40), and Δ is proportional to $J_0(\kappa' r)$ where

$$\kappa'^2 = \rho p^2/(\lambda + 2\mu) - \gamma^2 \quad\text{.................(43).}$$

To satisfy the equations

$$\left.\begin{aligned}\Delta &= \left\{\frac{dU}{dr} + \frac{U}{r} + \iota\gamma W\right\} e^{\iota(\gamma z + pt)},\\ 2\varpi_2 &= \left\{\iota\gamma U - \frac{dW}{dr}\right\} e^{\iota(\gamma z + pt)},\end{aligned}\right\}$$

we take

$$\left.\begin{aligned}U &= A\frac{d}{dr}J_0(\kappa' r) + B\gamma J_1(\kappa r),\\ W &= \iota\left[A\gamma J_0(\kappa' r) + \frac{B}{r}\frac{d}{dr}\left\{rJ_1(\kappa r)\right\}\right]\end{aligned}\right\}\text{... (44)};$$

and we find

$$\Delta = -A\,J_0(\kappa' r)\frac{\rho p^2}{\lambda + 2\mu},$$

$$2\varpi_2 = \iota B J_1(\kappa r)\frac{\rho p^2}{\mu}.$$

The conditions at the cylindrical boundary become

$$-\frac{\rho p^2\lambda}{\lambda + 2\mu}AJ_0(\kappa' r) + 2\mu\left\{A\frac{d^2}{dr^2}J_0(\kappa' r) + B\gamma\frac{d}{dr}J_1(\kappa r)\right\} = 0,$$

$$2\gamma A\frac{d}{dr}J_0(\kappa' r) + B\left\{\gamma^2 J_1(\kappa r) + \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\{rJ_1(\kappa r)\}\right]\right\} = 0,$$

when $r = c$.

From these, on eliminating A and B, we find the equation for the frequency

$$\left[2\mu\frac{d^2}{dc^2}J_0(\kappa' c) - \frac{\rho p^2\lambda}{\lambda + 2\mu}J_0(\kappa' c)\right]\left(2\gamma^2 - \frac{\rho p^2}{\mu}\right)J_1(\kappa c)$$
$$-4\mu\gamma^2\frac{d}{dc}J_1(\kappa c)\frac{d}{dc}J_0(\kappa' c) = 0 \quad\text{............(45),}$$

in which we have used the differential equation for $J_1(\kappa r)$.

Now writing

$$J_0(\kappa' c) = 1 - \frac{\kappa'^2 c^2}{2^2} + \frac{\kappa'^4 c^4}{2^2 . 4^2} - \dots,$$

$$J_1(\kappa c) = \kappa c\left(1 - \frac{\kappa^2 c^2}{2.4} + \dots\right),$$

we shall find that c is a factor, and, on removing this factor and leaving out all terms containing c, we obtain ultimately

$$\frac{p^2}{\gamma^2} = \frac{\mu}{\rho} \frac{3\lambda + 2\mu}{\lambda + \mu}.$$

Hence $$p = \gamma \sqrt{(E/\rho)} \qquad \text{(46)},$$

where E is Young's modulus. This gives the same rate of propagation as that found for extensional waves in art. 263.

If terms in c^2 be retained it will be found that, to a second approximation,

$$p = \gamma \sqrt{(E/\rho)} \left(1 - \frac{\sigma^2\gamma^2c^2}{4}\right) \qquad \text{(47)},$$

where σ is Poisson's ratio $\frac{1}{2}\lambda/(\lambda + \mu)$.

This result was first given by Prof. Pochhammer in *Crelle-Borchardt*, LXXXI. 1876, and afterwards apparently independently by Mr Chree in the *Quarterly Journal*, 1886. The latter writer has generalised it[1] for any form of section, and for an æolotropic rod whose elastic central-line is an axis of material symmetry, shewing that the constants we have here called p and γ are connected by the relation

$$p = \gamma \sqrt{(E/\rho)} \{1 - \tfrac{1}{2}\sigma^2\gamma^2k^2\} \qquad \text{(48)},$$

where σ is the Poisson's ratio of the material for extension in the direction of the axis of the rod, and k is the radius of gyration of the normal section about this axis.

271. Terminal conditions.

The solution just obtained is the exact solution for extensional vibrations of a circular cylinder, and it shews that when these vibrations are being executed there is lateral motion of every element of the cylinder not initially upon its axis, and the lateral displacement at a distance r from the axis is, in a very thin cylinder, proportional to r.

To find the conditions that obtain at the ends we form the

[1] 'On the longitudinal vibrations of æolotropic bars....' *Quarterly Journal*, 1890.

expressions for the stresses $\widehat{zz}$, $\widehat{zr}$, $\widehat{z\theta}$ across a normal section. The stress $\widehat{z\theta}$ vanishes, and the other two are

$$\widehat{zz} = \left[\left\{-AJ_0(\kappa' r)\left\{\frac{\rho p^2\lambda}{\lambda+2\mu}+2\mu\gamma^2\right\}-2\mu\gamma\frac{B}{r}\frac{d}{dr}\left[rJ_1(\kappa r)\right]\right\}\right]e^{\iota(\gamma z+pt)},$$

$$\widehat{zr} = 2\mu\left[\gamma A\frac{d}{dr}J_0(\kappa' r)+B\left\{\gamma^2 J_1(\kappa r)+\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left\{rJ_1(\kappa r)\right\}\right]\right\}\right]e^{\iota(\gamma z+pt)}.$$

Suppose l is the length of the rod, and z is measured from one end, then for a rod free at both ends we have to replace $e^{\iota\gamma z}$ by $\sin n\pi z/l$, where n is an integer. The solution will become real if A and B be pure imaginaries, and the real part be retained. Thus $v = 0$, and

$$\left.\begin{aligned} u &= \left[A_n\frac{d}{dr}J_0(\kappa_n' r)+\frac{n\pi}{l}B_nJ_1(\kappa_n r)\right]\sin\frac{n\pi z}{l}\cos(p_nt+\epsilon),\\ w &= \left[\frac{n\pi}{l}A_nJ_0(\kappa_n' r)+\frac{B_n}{r}\frac{d}{dr}\{rJ_1(\kappa_n r)\}\right]\cos\frac{n\pi z}{l}\cos(p_nt+\epsilon)\end{aligned}\right\}\quad(49)$$

are equations expressing a possible mode of free vibration of a cylinder of length l and radius c with both ends free, provided p_n is a certain function of $n\pi/l$, λ, μ, ρ, and c determined by a transcendental equation, while

$$\kappa_n^2 = p^2\rho/\mu - n^2\pi^2/l^2, \quad\text{and}\quad \kappa_n'^2 = p^2\rho/(\lambda+2\mu) - n^2\pi^2/l^2,$$

and further $A_n : B_n$ is a determinate ratio depending on c. When c is small such modes are ultimately identical with the "extensional" modes of a thin rod.

272. Flexural Vibrations.

The modes of vibration of the cylinder that correspond to flexural vibrations of a thin rod are more difficult of investigation, but they can be included among the modes under discussion by taking $\beta = 1$. We suppose then that

$$\left.\begin{aligned} u &= e^{\iota(\theta+\gamma z+pt)}\,U,\\ v &= e^{\iota(\theta+\gamma z+pt)}\,V,\\ w &= e^{\iota(\theta+\gamma z+pt)}\,W\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

and seek to determine U, V, W as functions of r.

By differentiation of the equations (35) we obtain the differential equation for Δ

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Delta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}+\frac{\partial^2\Delta}{\partial z^2}=\frac{\rho}{\lambda+2\mu}\frac{\partial^2\Delta}{\partial t^2}\quad\ldots\ldots\ldots(51).$$

[Cf. I. p. 310, equation (5).]

Since Δ is proportional to $e^{\iota(\theta+\gamma z+pt)}$ this equation becomes

$$\frac{\partial^2\Delta}{\partial r^2}+\frac{1}{r}\frac{\partial\Delta}{\partial r}+\left(\frac{\rho p^2}{\lambda+2\mu}-\gamma^2-\frac{1}{r^2}\right)\Delta=0\,\ldots\ldots\ldots\,(52),$$

which shews that Δ is proportional to $J_1(\kappa' r)$, κ' being the quantity defined by equation (43).

Now, eliminating Δ from the differential equations (35), we find

$$\left.\begin{aligned}\frac{\rho}{\mu}\frac{\partial^2\varpi_1}{\partial t^2}&=-\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{1}{r}\frac{\partial}{\partial r}(r\varpi_2)\right\}+\frac{1}{r^2}\frac{\partial^2\varpi_1}{\partial\theta^2}+\frac{\partial^2\varpi_1}{\partial z^2}-\frac{\partial}{\partial r}\left(\frac{\partial\varpi_3}{\partial z}\right),\\ \frac{\rho}{\mu}\frac{\partial^2\varpi_3}{\partial t^2}&=-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varpi_1}{\partial z}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varpi_3}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\varpi_3}{\partial\theta}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\varpi_2}{\partial z}\right)\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots(53).$$

From the second of these we can eliminate ϖ_1 and ϖ_2 by means of the identity (37), and we find

$$\frac{\rho}{\mu}\frac{\partial^2\varpi_3}{\partial t^2}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varpi_3}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\varpi_3}{\partial\theta^2}+\frac{\partial^2\varpi_3}{\partial z^2}\,\ldots\ldots\ldots\,(54).$$

Eliminating ϖ_2 from the first of (53) by means of the same identity we find

$$\frac{\rho}{\mu}\frac{\partial^2\varpi_1}{\partial t^2}=\frac{2}{r}\frac{\partial\varpi_3}{\partial z}+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}(r\varpi_1)\right\}+\frac{1}{r^2}\frac{\partial^2\varpi_1}{\partial\theta^2}+\frac{\partial^2\varpi_1}{\partial z^2}\,\ldots\,(55).$$

Now, remembering that ϖ_3 is proportional to $e^{\iota(\theta+\gamma z+pt)}$, we easily find from (54) that it is also proportional to $J_1(\kappa r)$, where κ is given by (40). Let us take

$$2\varpi_3=-\iota\kappa^2 CJ_1(\kappa r)e^{\iota(\theta+\gamma z+pt)}\,\ldots\ldots\ldots\ldots\ldots\ldots(56).$$

To satisfy equation (55), in which ϖ_1 is proportional to $e^{\iota(\theta+\gamma z+pt)}$, we have to take a complementary function, which satisfies the same equation when ϖ_3 is omitted, and any particular integral. It can be easily verified that, with the above value of ϖ_3, one particular integral is given by $2\varpi_1=C\gamma\dfrac{dJ_1(\kappa r)}{dr}$, while the complementary function is proportional to $\dfrac{1}{r}J_1(\kappa r)$.

We have now the forms of Δ, ϖ_1 and ϖ_2 expressed in terms

of Bessel's functions. To satisfy the equations connecting u, v, and w with Δ, ϖ_1, and ϖ_3 we assume

$$\left.\begin{aligned} U &= \left[A\frac{d}{dr}J_1(\kappa' r) + B\gamma\frac{d}{dr}J_1(\kappa r) + \frac{C}{r}J_1(\kappa r)\right], \\ V &= \iota\left[\frac{A}{r}J_1(\kappa' r) + \frac{B\gamma}{r}J_1(\kappa r) + C\frac{d}{dr}J_1(\kappa r)\right], \\ W &= \iota\left[A\gamma J_1(\kappa' r) - B\kappa^2 J_1(\kappa r)\right] \end{aligned}\right\}\ldots(57);$$

and it is easy to verify that

$$\left.\begin{aligned} \Delta &= -\frac{\rho p^2}{\lambda + 2\mu}AJ_1(\kappa' r)\,e^{\iota(\theta+\gamma z+pt)}, \\ 2\varpi_3 &= -\iota\kappa^2 CJ_1(\kappa r)e^{\iota(\theta+\gamma z+pt)}, \\ 2\varpi_1 &= \left[C\gamma\frac{dJ_1(\kappa r)}{dr} + B\frac{\rho p^2}{\mu}\frac{J_1(\kappa r)}{r}\right]e^{\iota(\theta+\gamma z+pt)} \end{aligned}\right\}\ldots(58).$$

The conditions at the cylindrical boundary are

$$\left.\begin{aligned} \lambda\Delta + 2\mu\frac{dU}{dr} &= 0, \\ \iota\frac{U}{r} + r\frac{d}{dr}\left(\frac{V}{r}\right) &= 0, \\ \iota\gamma U + \frac{dW}{dr} &= 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(59)$$

when $r = c$. These are linear in A, B, C, and if we eliminate A, B, C we shall obtain an equation connecting p, γ, ρ, λ, μ. This is the frequency-equation.

The general frequency-equation is very complicated, but if we suppose c small, and substitute for the Bessel's functions their expressions in series of ascending powers of c, and in the equation keep only the lowest powers of c, we shall find that p and c are factors, and, on removing them, the equation can be reduced to

$$p^2 = \frac{1}{4}\frac{E}{\rho}\gamma^4 c^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60),$$

where E is Young's modulus $\mu(3\lambda + 2\mu)/(\lambda + \mu)$.

It follows that each of the displacements satisfies an equation of the form

$$-\frac{1}{4}c^2\frac{E}{\rho}\frac{\partial^4 u}{\partial z^4} = \frac{\partial^2 u}{\partial t^2}\ldots\ldots\ldots\ldots\ldots\ldots(61),$$

so that the vibrations in question are ultimately identical with those considered in art. 265.

The above is the exact solution for those modes of vibration of a circular cylinder which approximate to the flexural vibrations usually considered in the case of a rod supposed very long and thin. It appears on inspection of the formulæ that the lateral displacement is accompanied by a displacement parallel to the axis at all points not initially on the axis of the cylinder.

273. Terminal conditions.

The discussion of the boundary-conditions at the ends of the cylinder is more difficult than in the preceding cases, and it is convenient to change to a real quantity in place of $\iota\gamma$. We have found that approximately

$$\gamma^4 = 4p^2\rho/Ec^2,$$

and, if we call the real positive fourth root of this quantity m/l, we can verify that there exists a solution of the form

$$\left.\begin{aligned}
u &= \left[A\frac{d}{dr}J_1(h'r)+B\frac{m}{l}\frac{d}{dr}J_1(hr)+\frac{C}{r}J_1(hr)\right]e^{mz/l}\,\frac{\sin}{\cos}\,\theta\cos(pt+\epsilon),\\
v &= \left[A\frac{1}{r}J_1(h'r)+B\frac{m}{l}\frac{1}{r}J_1(hr)+C\frac{d}{dr}J_1(hr)\right]e^{mz/l}\,\frac{\cos}{-\sin}\,\theta\cos(pt+\epsilon),\\
w &= \left[A\frac{m}{l}J_1(h'r)-Bh^2J_1(hr)\right]e^{mz/l}\,\frac{\sin}{\cos}\,\theta\cos(pt+\epsilon)
\end{aligned}\right\}\quad\text{..................(62)},$$

where $\quad h^2 = p^2\rho/\mu + m^2/l^2$, and $h'^2 = p^2\rho/(\lambda+2\mu) + m^2/l^2$...(63),

and the ratios of $A : B : C$ are determinate from the boundary-conditions that hold at the curved surface of the cylinder.

There exist like solutions in $e^{-mz/l}$, the sign of m/l being changed throughout.

There also exist solutions in simple harmonic functions of mz/l. Of these one can be obtained from the above by writing $\cos mz/l$ instead of $e^{mz/l}$ in u and v, and $-\sin mz/l$ in w, and at the same time putting κ^2 and κ'^2 in place of h^2 and h'^2. The other will be obtained by writing $\sin mz/l$ instead of $e^{mz/l}$ in u and v, and $\cos mz/l$ in w, and at the same time putting κ^2 and κ'^2 in place of h^2 and h'^2.

Taking the case where both ends are free we have

$$\left.\begin{aligned} \lambda\Delta + 2\mu\frac{\partial w}{\partial z} &= 0, \\ \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta} &= 0, \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} &= 0 \end{aligned}\right\} \qquad \text{(64)}$$

at $z = 0$ and at $z = l$.

It is not generally possible to satisfy these six conditions by any linear combination of the four forms of solution obtained, in other words, the solutions do not give a system of normal functions for free vibrations of a cylinder of finite thickness and finite length; but when the cylinder is very thin the third of these conditions is approximately satisfied by each of the forms. It is in fact identically satisfied when $r = c$; and, c being supposed small, the condition $\widehat{zr} = 0$ when $r = c$ gives rise to equations of the form

$$2A\gamma h' + B(h^2 + \gamma^2)\,h + C\gamma h + [\text{terms involving } c^2] = 0\,;$$

so that, if the terms involving c^2 be rejected, the stress $\widehat{zr}$ is identically zero at all points of the cylinder.

If we simplify the boundary-conditions (64) by omitting the last of them, $\widehat{zr} = 0$, the remaining four give rise to four relations connecting the four arbitrary constants and containing e^m, e^{-m}, $\sin m$ and $\cos m$, and on elimination of the constants we should obtain an equation to determine m. The work would be very complicated, and, after what we have said about the normal functions and period equation for flexural vibrations of a thin rod, it is not worth while to go through it. But it is important to notice that the boundary-conditions at free ends cannot be satisfied exactly in the case of these modes as they can in the torsional and extensional modes.

We might investigate other modes of vibration possible in an infinite cylinder by putting $\beta = 2, 3, \ldots$ and supposing γ real, but they would have no reference to thin vibrating rods under ordinary conditions.

CHAPTER XVII.

RESILIENCE AND IMPACT.

274. THE term "resilience" was introduced into Physics by Thomas Young[1] with the definition:—"The action which resists "pressure is called strength, and that which resists impulse may "properly be termed resilience... The resilience of a body is "jointly proportional to its strength and its toughness, and is "measured by the product of the mass and the square of the "velocity of a body capable of breaking it, or of the mass and the "height from which it must fall in order to acquire that velocity; "while the strength is merely measured by the greatest pressure "which it can support". The word has been variously used for the work done in producing rupture, the potential energy of the greatest strain of a given type possible within the elastic limits, and to express a certain property of matter. It is in the last of these senses that we shall use it. We may regard Young's definition as asking a question:—What is the strain produced by a given body striking a given body in a given manner at a given point with a given velocity?—and we may generalise the question, and enquire what is the strain produced in a given body when small internal relative motions are set up in it by the action of given forces, or by the impact or sudden attachment of other given bodies?

The property of matter which we term resilience depends on the same circumstances as that which French writers call *résistance vive* or dynamical resistance, and its meaning can be best

[1] *A Course of Lectures on Natural Philosophy and the Mechanical Arts*, vol. I. p. 143. London, 1807.

brought out by comparing that of statical resistance. A given elastic solid in equilibrium under a given load is in a certain state of strain. But, if the same load were applied to the same solid in any other state, motion would be set up, and a different strain would be produced. Some extreme cases have been considered in I. art. 80, where it was shewn that sudden applications and sudden reversals of load are attended by strains generally much in excess of the statical strains. The resistance of the solid to strain produced by a given load is correspondingly diminished.

We can now give as a general definition:—Resilience is a property of matter in virtue of which the strain produced in a body depends on the *motions* set up in the body and on the *motion* of the apparatus applying the load.

The theory of the vibrations of thin rods or bars has put us in possession of a description of certain possible modes of small motion of elastic bodies which admit of simple discussion, and we shall proceed to consider some elementary examples in which the resilience of bars comes into play. One of these, viz. the problem of the longitudinal impact of two bars, will lead us to the discussion of theories of impact. The subjects of impact and resilience are connected, but resilience is important in many cases in which there is no impulsive action. To adopt the ordinary terminology of Rational Mechanics it has to do with 'initial motions' and 'small oscillations' as well as with 'impulses'.

275. Bar struck longitudinally, one terminal fixed[1].

We shall consider in the first place a bar fixed at one end and free at the other, and we shall suppose that a body of mass M moving in the line of the bar with velocity V strikes it at its free end. The impulse tends to produce compression in the bar, and the motion produced must be in accordance with the differential equation of extensional vibrations.

Let the origin be at the free end of the unstrained bar, let the axis z be along the central line of the bar, and let l be the length and ω the cross-section of the bar, ρ the density and E the Young's modulus of the material, and let w be the displacement

[1] Boussinesq, *Applications des Potentiels...*; also Saint-Venant, 'Annotated Clebsch', *Note finale du* § 60 and *Changements et Additions*, and Pearson's *Elastical Researches of Barré de Saint-Venant*, arts. 339—341 and 401—407.

of any point on the elastic central-line in the direction of the axis z; then the motion of the bar is governed by the equation

$$\rho \frac{\partial^2 w}{\partial t^2} = E \frac{\partial^2 w}{\partial z^2} \quad \text{.........................} \quad (1).$$

If $E = a^2 \rho$, so that a is the velocity of propagation of extensional waves, the general solution of the equation may be written

$$w = f(at - z) + F(at + z) \quad \text{..................} \quad (2),$$

where f and F denote arbitrary functions.

The bar being fixed at $z = l$ we have

$$f(at - l) + F(at + l) = 0$$

for all values of t. Hence for any argument ζ

$$F(\zeta) = -f(\zeta - 2l)$$

and the equation (2) therefore becomes

$$w = f(at - z) - f(at + z - 2l) \quad \text{..................} \quad (3).$$

So long as the body M and the end of the bar remain in contact the pressure between them is $-(E \partial w/\partial z)_{z=0}$, and the acceleration of the body is $[\partial^2 w/\partial t^2]_{z=0}$, so that the equation of motion of the body is

$$\left[M \frac{\partial^2 w}{\partial t^2}\right]_{z=0} = E \left[\frac{\partial w}{\partial z}\right]_{z=0} \quad \text{..................} \quad (4),$$

and this is the terminal condition at $z = 0$.

This condition is

$$f''(at) - f''(at - 2l) = \frac{E\omega}{Ma^2}[-f'(at) - f'(at - 2l)].$$

Since E is $a^2 \rho$, the multiplier $E\omega/Ma^2$ on the right becomes $1/ml$, where m is the ratio (mass of striking body) : (mass of rod), and the condition becomes

$$f''(at) + \frac{1}{ml} f'(at) = f''(at - 2l) - \frac{1}{ml} f'(at - 2l) \quad \text{......} \quad (5).$$

To integrate this equation we must introduce the conditions that hold when $t = 0$ or has a negative value.

If t be negative and $l > z > 0$, w, $\partial w/\partial t$, and $\partial w/\partial z$ are all zero, and we find that $f(\zeta)$ and $f'(\zeta)$ both vanish for all negative values of the argument ζ.

If t be zero $f(\zeta)$ vanishes, but $f'(\zeta)$ does not vanish. In fact $\partial w/\partial t$ is zero for all values of z except $z=0$, and $(\partial w/\partial t)_{z=0,\ t=0}=V$, the velocity of the impinging body, since the element of the bar at $z=0$ takes this velocity impulsively.

Thus we have $$af'(at)_{t=0}=V$$

or $$f'(\zeta)_{\zeta=0}=V/a.$$

276. The Continuing Equation[1].

Let us now integrate equation (5) and put when $t=0$ $f'(at)=V/a$, and $f(at)$, $f'(at-2l)$, $f(at-2l)$ all zero; thus we have

$$f'(at)+\frac{1}{ml}f(at)=\frac{V}{a}+f'(at-2l)-\frac{1}{ml}f(at-2l).$$

In terms of an argument ζ this becomes

$$f'(\zeta)+\frac{1}{ml}f(\zeta)=\frac{V}{a}+f'(\zeta-2l)-\frac{1}{ml}f(\zeta-2l)\text{......(6)}.$$

If we suppose the right-hand member known this is a linear differential equation to determine f, and, on integrating it, we have

$$f(\zeta)=e^{-\zeta/ml}\int_0^\zeta e^{\zeta/ml}\left[V/a+f'(\zeta-2l)-f(\zeta-2l)/ml\right]d\zeta\text{ ...(7)},$$

where the condition that $f(\zeta)$ vanishes with ζ has been introduced. Now in the integral on the right we can suppose $f'(\zeta-2l)$ and $f(\zeta-2l)$ known. For when $2l>\zeta>0$ they vanish, and thus $f(\zeta)$ and consequently $f'(\zeta)$ are determined for this interval. By integration we may then determine $f(\zeta)$ for the interval in which $4l>\zeta>2l$, and $f'(\zeta)$ can be found by differentiation. Proceeding in this way the equation (7) enables us to determine the values of $f(\zeta)$ and $f'(\zeta)$ in any interval $(2n+2)l>\zeta>2nl$, when their values in the previous interval have been found. We call this equation the *continuing equation*[2].

[1] The method of solution of problems on extensional vibrations in terms of discontinuous functions consists generally of three steps, (1) the determination of the function for a certain range of values of the variable by means of the initial conditions, (2) the formation of a continuing equation for deducing the values of the function for other values of the variable, (3) the solution of the continuing equation.

[2] *Équation promotrice* of Saint-Venant.

From the continuing equation we find

when $2l > \zeta > 0$

$$f(\zeta) = ml\frac{V}{a}(1 - e^{-\zeta/ml}),\ f'(\zeta) = \frac{V}{a}e^{-\zeta/ml};$$

when $4l > \zeta > 2l$

$$f(\zeta) = ml\frac{V}{a}(1 - e^{-\zeta/ml}) + ml\frac{V}{a}\left[-1 + \left(1 + 2\frac{\zeta - 2l}{ml}\right)e^{-(\zeta-2l)/ml}\right],$$

$$f'(\zeta) = \frac{V}{a}e^{-\zeta/ml} + \frac{V}{a}\left(1 - 2\frac{\zeta - 2l}{ml}\right)e^{-(\zeta-2l)/ml};$$

when $6l > \zeta > 4l$

$$f(\zeta) = ml\frac{V}{a}(1 - e^{-\zeta/ml}) + ml\frac{V}{a}\left[-1 + \left(1 + 2\frac{\zeta - 2l}{ml}\right)e^{-(\zeta-2l)/ml}\right]$$
$$+ ml\frac{V}{a}\left[1 - \left\{1 + 2\left(\frac{\zeta - 4l}{ml}\right)^2\right\}e^{-(\zeta-4l)/ml}\right],$$

$$f'(\zeta) = \frac{V}{a}e^{-\zeta/ml} + \frac{V}{a}\left(1 - 2\frac{\zeta - 2l}{ml}\right)e^{-(\zeta-2l)/ml}$$
$$+ \frac{V}{a}\left[1 - 4\frac{\zeta - 4l}{ml} + 2\left(\frac{\zeta - 4l}{ml}\right)^2\right]e^{-(\zeta-4l)/ml};$$

....

277. Circumstances of the impact.

The impact terminates when the sign of $(\partial w/\partial z)_{z=0}$ changes, for the body M is unable to exert a tension on the end $z = 0$. From this instant onwards the above solution ceases to hold, and the bar vibrates with one end fixed and the other free.

Now $$-(\partial w/\partial z)_{z=0} = f'(at) + f'(at - 2l) \qquad \text{............(8).}$$

When $2l > at > 0$ the second term vanishes, and the first is $\frac{V}{a}e^{-\zeta/ml}$, which does not change sign. Hence the impact cannot terminate before $at = 2l$.

The impact terminates in the interval $4l > at > 2l$ if

$$\frac{V}{a}e^{-at/ml}\left[1 + \left(1 - 2\frac{at - 2l}{ml}\right)e^{2/m} + e^{2/m}\right]$$

can vanish for a value t between $4l/a$ and $2l/a$.

The value of t for which it vanishes is given by the equation

$$\frac{2at}{ml} = \frac{4}{m} + 2 + e^{-2/m} \text{ ....(9)},$$

and the condition that $t < 4l/a$ gives

$$2 + e^{-2/m} < 4/m.$$

It is easy to see that the equation $2 + e^{-2/m} = 4/m$ has a ro lying between $m = 1$ and $m = 2$ and by calculation the root found to be

$$m = 1{\cdot}7283.$$

Hence if the ratio (mass of striking body) : (mass of rod) b $< 1{\cdot}7283$ the impact terminates in the interval between $t = 2l/$ and $t = 4l/a$, and the time at which it terminates is given b equation (9).

When $m < 1{\cdot}7283$ the velocity with which the body M rebound is easily shewn to be

$$2Ve^{-(1+\frac{1}{2}e^{-2/m})}.$$

If m be $> 1{\cdot}7283$ the duration of impact must exceed $4l/$ To find the condition that the impact terminates between $t = 4l/$ and $t = 6l/a$ we have the equation

$$1 + \left(1 - 2\frac{at - 2l}{ml}\right)e^{2/m} + \left[1 - 4\frac{at - 4l}{ml} + 2\left(\frac{at - 4l}{ml}\right)^2\right]e^{4/m}$$
$$+ e^{2/m} + \left(1 - 2\frac{at - 2l}{ml}\right)e^{4/m} = 0 \text{ ....(10)},$$

of which the root at must be $< 6l$, so that the extreme value of is given by the equation

$$1 + \left(2 - \frac{8}{m}\right)e^{2/m} + \left(2 - \frac{16}{m} + \frac{8}{m^2}\right)e^{4/m} = 0.$$

The root can be shewn to be

$$m = 4{\cdot}1511.$$

Hence if $4{\cdot}1511 > m > 1{\cdot}7283$ the impact terminates in th interval between $t = 4l/a$ and $t = 6l/a$, and the time at which i terminates is given by equation (10).

For further details the reader is referred to the authoritie quoted on p. 126.

It is proved *inter alia* that the maximum compression take place at the fixed end[1], and if $m < 5$ its value is $2(1 + e^{-2/m})\,V/$

[1] This point can be at once established by observing that the compression the fixed end is $2f'(at - l)$ at time t. The compression at a point distant z fro the end struck is $f'(at - z) + f'(at + z - 2l)$. Whichever is the greater of these tw terms the sum of them is less than twice the greater, and there will at some tim exist at the fixed end a compression equal to twice the greater.

but if $m > 5$ the value is approximately $(1 + \sqrt{m})\, V/a$. As equal extensions will take place after the end struck becomes free the limiting velocity consistent with safety, according to the theory of Poncelet and Saint-Venant (I. p. 107), will be given by the equations

$$\Phi V = \tfrac{1}{2}\frac{T_0}{E} a \frac{1}{1+e^{-2/m}} \text{ when } m < 5$$

and
$$\Phi V = \frac{T_0}{E} a \frac{1}{1+\sqrt{m}} \text{ when } m > 5,$$

where T_0 is the breaking tension for pull in the direction of the elastic central-line of the bar, and Φ is the factor of safety.

If the problem were treated as a statical problem by the neglect of the inertia of the bar we should find the greatest strain equal to $\sqrt{m}\, V/a$, so that the effect of this inertia is to diminish the limiting safe velocity.

278. Bar struck longitudinally, terminals free[1].

When the end $z = l$ of the bar is free we take, to satisfy the terminal condition that $\partial w/\partial z = 0$ when $z = l$ for all values of t,

$$w = f(at - z) + f(at + z - 2l) \ldots\ldots\ldots\ldots\ldots (11),$$

and the terminal condition at $z = 0$ corresponding to (5) becomes

$$f''(at) + \frac{1}{ml} f'(at) = -\left[f''(at - 2l) - \frac{1}{ml} f'(at - 2l)\right].$$

As before we find

$$f'(\zeta) + \frac{1}{ml} f(\zeta) = \frac{V}{a} - f'(\zeta - 2l) + \frac{1}{ml} f(\zeta - 2l),$$

leading to the continuing equation

$$f(\zeta) = e^{-\zeta/ml} \int_0^{\zeta} \{V/a - f'(\zeta - 2l) + f(\zeta - 2l)/ml\}\, e^{\zeta/ml} d\zeta \ldots (12).$$

When $2l > \zeta > 0$ this gives

$$f(\zeta) = ml \frac{V}{a}(1 - e^{-\zeta/ml}), \quad f'(\zeta) = \frac{V}{a} e^{-\zeta/ml} \ldots\ldots (13);$$

when $4l > \zeta > 2l$

$$f(\zeta) = ml \frac{V}{a}(1 - e^{-\zeta/ml}) - ml \frac{V}{a}\left[-1 + \left(1 + 2\frac{\zeta - 2l}{ml}\right) e^{-(\zeta - 2l)/ml}\right],$$

$$f'(\zeta) = \frac{V}{a} e^{-\zeta/ml} - \frac{V}{a}\left(1 - 2\frac{\zeta - 2l}{ml}\right) e^{-(\zeta - 2l)/ml}.$$

[1] Boussinesq, *loc. cit.*

When $2l > at > 0$ we find

$$-(\partial w/\partial z)_0 = e^{-at/ml} V/a,$$

so that the impact does not terminate before $t = 2l/a$.

When $4l > at > 2l$ we find

$$-(\partial w/\partial z)_0 = \frac{V}{a} e^{-at/ml} \left[1 - e^{2/m}\left(1 - 2\frac{at-2l}{ml}\right) - e^{2/m}\right].$$

When $at = 2l$ this contains as a factor $1 - 2e^{2/m}$, which is negative, so that $(\partial w/\partial z)_0$ always changes sign when $at = 2l$, and the impact always terminates after the time taken by an extensional wave to travel twice the length of the bar.

The velocity of the mass M immediately before the termination of the impact is the value of

$$a[f'(at) + f'(at - 2l)]$$

when $at = 2l$ (or rather is just less than $2l$), and the first term is to be found from (13) while the second term vanishes. Hence the velocity in question is $Ve^{-2/m}$. Thus the mass M proceeds in the same direction with reduced velocity.

The bar is now free and its state as regards compression and velocity are expressed by the equations

$$\left.\begin{aligned}\frac{\partial w}{\partial z} &= -\frac{V}{a} e^{-(2l-z)/ml} + \frac{V}{a} e^{-z/ml},\\ \frac{\partial w}{\partial t} &= Ve^{-(2l-z)/ml} + Ve^{-z/ml}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(14),$$

together with $$\frac{\partial w}{\partial z} = 0 \text{ when } z = 0.$$

This last gives rise to a new continuing equation

$$-f'(at) + f'(at - 2l) = 0,$$

or $$f'(\zeta) = f'(\zeta - 2l) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15),$$

which holds for all values of ζ.

Now measuring t from the instant when the impact terminates we have from (14), when $l > z > 0$,

$$-f'(-z) + f'(z - 2l) = \frac{V}{a}[e^{-z/ml} - e^{-(2l-z)/ml}],$$

$$f'(-z) + f'(z - 2l) = \frac{V}{a}[e^{-z/ml} + e^{-(2l-z)/ml}].$$

Hence when $l > z > 0$

$$f'(-z) = \frac{V}{a} e^{(2l-z)/ml},$$

$$f'(z-2l) = \frac{V}{a} e^{-z/ml}.$$

Thus when $-l > \zeta > -2l$

$$f'(\zeta) = \frac{V}{a} e^{-(2l+\zeta)/ml} \quad \text{.....................(16)},$$

and when $0 > \zeta > -l$

$$f'(\zeta) = \frac{V}{a} e^{(2l+\zeta)/ml} \quad \text{.....................(17)},$$

and the continuing equation (15) will enable us to find the value of $f'(\zeta)$ as long as the solution continues to hold.

The velocity of the end $z = 0$ at time t is

$$a[f'(at) + f'(at - 2l)]$$

or, by the continuing equation, $2af'(at - 2l)$.

When $l > at > 0$ this is $2Ve^{-at/ml}$,

when $2l > at > l$ it is $2Ve^{at/ml}$.

When at is increased by $2l$ the value of the velocity recurs.

Now the velocity of the mass M after the termination of the impact has been found to be $Ve^{-2/m}$, while the velocity of the end $z=0$ is never less than $2Ve^{-1/m}$, its value at time $t = l/a$. Hence after the impact the mass M has always a smaller velocity than the nearer end of the rod, and the two bodies never again impinge. It follows that the solution expressed by (16) and (17) and the continuing equation (15) continues to hold indefinitely.

279. Bar suddenly loaded.

The method of the preceding problems is also applicable to the case of an upright column supporting a weight.

We shall suppose that a mass M is gently placed on the top of a straight vertical bar, so that initially no part of the system has any velocity and the initial compression in the bar is that produced by its own weight, and we shall suppose the lower end of the bar fixed.

Measuring z from the upper end of the bar, the differential equation is

$$\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial z^2} + g \quad \text{.....................(18)},$$

and if we put $$w = \tfrac{1}{2} g (l^2 - z^2)/a^2 + w' \quad \text{..................(19)},$$

w' satisfies the equation $$\frac{\partial^2 w'}{\partial t^2} = a^2 \frac{\partial^2 w'}{\partial z^2},$$

and to satisfy the condition at the fixed end, $z = l$, we have to take

$$w' = \phi(at - z) - \phi(at + z - 2l) \quad \text{............(20)}.$$

The condition at the end $z = 0$ is

$$M \left[\frac{\partial^2 w}{\partial t^2}\right]_{z=0} = Mg + E\omega \left[\frac{\partial w}{\partial z}\right]_{z=0},$$

and, with the notation of the two preceding problems, this gives us

$$\phi''(\zeta) + \frac{1}{ml}\phi'(\zeta) = \frac{g}{a^2} + \left[\phi''(\zeta - 2l) - \frac{1}{ml}\phi'(\zeta - 2l)\right] \quad (21),$$

in which at has been replaced by ζ.

As in art. 275 we can shew that $\phi'(\zeta)$ and $\phi(\zeta)$ vanish for zero and negative values of ζ, and writing (21) in terms of an argument ζ and integrating we have

$$\phi'(\zeta) + \frac{1}{ml}\phi(\zeta) = \frac{g\zeta}{a^2} + \left[\phi'(\zeta - 2l) - \frac{1}{ml}\phi(\zeta - 2l)\right].$$

Whence

$$\phi(\zeta) = e^{-\zeta/ml} \int_0^\zeta e^{\zeta/ml} \left[\frac{g}{a^2}\zeta + \phi'(\zeta - 2l) - \frac{1}{ml}\phi(\zeta - 2l)\right] d\zeta \text{...(22)}.$$

Hence when $2l > \zeta > 0$ we deduce

$$\left.\begin{aligned} \phi(\zeta) &= \frac{g}{a^2} m^2 l^2 \left[\frac{\zeta}{ml} - 1 + e^{-\zeta/ml}\right], \\ \phi'(\zeta) &= \frac{g}{a^2} ml \left[1 - e^{-\zeta/ml}\right] \end{aligned}\right\} \quad \text{.........(23)}.$$

We observe that when $2l > \zeta > 0$, $\phi'(\zeta)$ in this problem can be obtained from $f(\zeta)$ of art. 276 by writing g/a for V, also that the equation (21) can be obtained from (6) of art. 276 by the same substitution. We conclude that for all values of ζ, $\phi'(\zeta)$ can be deduced from $f(\zeta)$ of art. 276 by changing V into g/a.

To determine the velocity or compression at any point of the bar at any time we require only to know $\phi'(\zeta)$ and therefore these quantities are known by the solution of the problem in art. 276.

It must be noticed that the solution is not to be stopped when $f'(at)+f'(at-2l)$ vanishes, but the method of art. 276 is to be continued indefinitely, as if in the problem there treated the mass M became attached to the extremity of the bar at the impact.

It is worth while to remark that if the bar be fixed at its upper end $z=l$, and support a weight at its other end $z=0$, which is suddenly, but not impulsively, attached to the bar, the solution will be the same except that the sign of g must be changed.

280. **Particular Cases.**

We proceed to give some arithmetical results in particular cases. The limiting case of m infinite has been considered in I. art. 81, where it was remarked that if a weight be suddenly attached to a weightless elastic string the greatest subsequent extension is twice what it would be if the weight were applied gradually, and the like is true for compression in a massless bar whose lower end is fixed and upper end loaded. We shall consider the cases of $m=1, 2, 4$, and it will be seen that the dynamical strain is in each case considerably greater than the statical, and the ratio for $m=4$ is very nearly equal to 2. In each case it may be observed that, for all values of ζ, $\phi'(\zeta)$ contains a factor mlg/a^2. We shall give the values of the compression $-(\partial w/\partial z)$, and the velocity $(\partial w/\partial t)$, at the loaded end. These are found from the formulæ

$$-\left(\frac{\partial w}{\partial z}\right)_{z=0}=\phi'(at)+\phi'(at-2l)$$

and
$$\frac{1}{a}\left(\frac{\partial w}{\partial t}\right)_{z=0}=\phi'(at)-\phi'(at-2l).$$

The value of the compression at the fixed end is $lg/a^2+2\phi'(at-l)$, and the maximum of this is the greatest compression. The calculations become very tedious as at/l increases, and we shall content ourselves with giving the first maximum.

When the bar has its upper end fixed the same calculations give us the *extension* at the loaded and fixed ends, and the velocity of the loaded end at any time not too great.

Case 1°. $m=1$.

All the values of $\phi'(\zeta)$ contain lg/a^2 as a factor. If we denote by e_0 the value of $-\frac{a^2}{lg}\left(\frac{\partial w}{\partial z}\right)_{z=0}$, and by v_0 the value of $\frac{a}{lg}\left(\frac{\partial w}{\partial t}\right)_{z=0}$, so that e_0 is a definite multiple of the terminal compression, and v_0 of the terminal velocity, we can find the following:—

$at/l=0$	1	2	3	4	5	6	7	8
$e_0=0$	·632	·865	1·686	1·622	1·610	1·203	·979	1·154
$v_0=0$	·632	·865	·422	−·107	−·498	−·311	−·132	·284

The first maximum value of $\phi'(\zeta)$ occurs when $\zeta/l=2·567$, and this value is 1·136. Hence the greatest compression experienced at the fixed end before $at=8l$ is

$$\frac{lg}{a^2}+2\frac{lg}{a^2}\phi'\{l(2·567)\}=\frac{lg}{a^2}(3·273)$$

and this strain occurs when $at/l=3·567$.

If the inertia of the bar were neglected, or the weight gradually applied, the greatest compression would be $2lg/a^2$, so that the first maximum compression exceeds the statical compression in the ratio 1·63 : 1.

In like manner for a bar supporting a weight equal to its own weight the greatest extension when the weight is suddenly attached is greater than that when the weight is gradually applied in the same ratio.

Case 2°. $m=2$.

All the values of $\phi'(\zeta)$ contain $2lg/a^2$ as a factor. If we denote by e_0 the value of $-\frac{a^2}{2lg}\left(\frac{\partial w}{\partial z}\right)_{z=0}$, and by v_0 that of $\frac{a}{2lg}\left(\frac{\partial w}{\partial t}\right)_{z=0}$, we can find the following:—

$at/l=0$	1	2	3	4	5	6	7	8
$e_0=0$	·393	·632	1·394	1·600	1·943	1·591	1·198	·594
$v_0=0$	·393	·632	·597	·336	−·038	−·345	−·706	−·652

The first maximum value of $\phi'(\zeta)$ occurs when $\zeta/l=3·368$, and this value is 1·019. Hence the greatest compression experienced at the fixed end before $at=8l$ is

$$\frac{lg}{a^2} + 2\frac{2lg}{a^2}\phi'\{l(3\cdot368)\} = \frac{lg}{a^2}(5\cdot077),$$

and this strain occurs when $at/l = 4\cdot368$.

The statical compression would be $3lg/a^2$, so that the dynamical maximum strain exceeds the statical in the ratio 1·7 : 1 nearly.

Case 3°. $m = 4$.

All the values of $\phi'(\zeta)$ contain $4lg/a^2$ as a factor. We find

$\zeta/l =$	0	1	2	3	4	5	6	7	8
$\frac{a^2}{4lg}\phi'(\zeta) =$	0	·221	·393	·696	·845	1·017	·971	·912	·664

The first maximum occurs when $\zeta/l = 5\cdot296$, and the corresponding value of $\phi'(\zeta)$ is 1·122. Hence the greatest compression experienced at the fixed end before $at = 8l$ is

$$\frac{lg}{a^2} + 2\frac{4lg}{a^2}\phi'\{l(5\cdot296)\} = \frac{lg}{a^2}(9\cdot973),$$

and this strain occurs when $at/l = 6\cdot296$.

In this case the dynamical strain exceeds the statical in the ratio 9·973 : 5 or 1·99 : 1.

281. Longitudinal Impact of Bars[1].

Another problem that can be solved by the same kind of analysis is the famous problem of the longitudinal impact of two bars.

We shall consider the case in which a bar of length l_1 moving with velocity V_1 overtakes a bar of length $l_2\ (> l_1)$ moving with velocity $V_2\,(< V_1)$, and we shall suppose the bars of the same material and section.

Suppose z_1 is measured along the bar (1) and z_2 along the bar (2) from the junction, and let w_1 be the displacement of a point on the central-line of the bar (1) parallel to this line, and w_2 that in the bar (2), measured in the same directions as z_1 and z_2. The differential equations of motion of the two bars are

[1] Saint-Venant, Liouville's *Journal*, XII. 1867. In this memoir the bars during the impact are treated as forming a single continuous bar, but this is unnecessary.

$$\left.\begin{aligned}\frac{\partial^2 w_1}{\partial t^2}&=a^2\frac{\partial^2 w_1}{\partial z^2},\\ \frac{\partial^2 w_2}{\partial t^2}&=a^2\frac{\partial^2 w_2}{\partial z^2}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24),$$

where a is the velocity of extensional waves in either bar.

The conditions at time $t=0$ are that, in the bar (1)

$$\frac{\partial w_1}{\partial t}=-V_1,\quad \frac{\partial w_1}{\partial z_1}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

and, in the bar (2)

$$\frac{\partial w_2}{\partial t}=V_2,\quad \frac{\partial w_2}{\partial z_2}=0\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

The conditions at the free ends are that

$$\left.\begin{aligned}\partial w_1/\partial z_1=0 \text{ when } z_1=l_1,\\ \partial w_2/\partial z_2=0 \text{ when } z_2=l_2\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

The conditions at the junction are that when z_1 and z_2 vanish

$$\frac{\partial w_1}{\partial t}+\frac{\partial w_2}{\partial t}=0,\quad \frac{\partial w_1}{\partial z_1}-\frac{\partial w_2}{\partial z_2}=0\ldots\ldots\ldots\ldots\ldots(28).$$

To satisfy the equations (24) and the conditions (27) we have to take

$$\left.\begin{aligned}w_1&=f(at-z_1)+f(at+z_1-2l_1),\\ w_2&=F(at-z_2)+F(at+z_2-2l_2)\end{aligned}\right\}\ldots\ldots\ldots\ldots(29).$$

The conditions at time $t=0$ give us

when $l_1>z_1>0$
$$-f'(-z_1)+f'(z_1-2l_1)=0,$$
$$f'(-z_1)+f'(z_1-2l_1)=-V_1/a,$$

and when $l_2>z_2>0$
$$-F'(-z_2)+F'(z_2-2l_2)=0,$$
$$F'(-z_2)+F'(z_2-2l_2)=V_2/a.$$

From these we have, in terms of an argument ζ,

$$\left.\begin{aligned}&\text{when } 0>\zeta>-2l_1 \quad f'(\zeta)=-\tfrac{1}{2}V_1/a,\\ &\text{when } 0>\zeta>-2l_2 \quad F'(\zeta)=\tfrac{1}{2}V_2/a\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

The conditions (28) at the junction give us

$$f'(at)+f'(at-2l_1)+F'(at)+F'(at-2l_2)=0,$$
$$-f'(at)+f'(at-2l_1)+F'(at)-F'(at-2l_2)=0;$$

and from these we have the continuing equations

$$\left.\begin{aligned}f'(\zeta)&=-F'(\zeta-2l_2),\\ F'(\zeta)&=-f'(\zeta-2l_1)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(31).$$

The solution continues to hold so long as $(\partial w_1/\partial z_1)_{z_1=0}$ and $(\partial w_2/\partial z_2)_{z_2=0}$ are both negative.

Now until $at = 2l_1$

$$(\partial w_1/\partial z_1)_{z_1=0} = (\partial w_2/\partial z_2)_{z_2=0} = -\tfrac{1}{2}(V_1 - V_2)/a.$$

When $\quad at > 2l_1 \quad (\partial w_1/\partial z_1)_{z_1=0} = 0,$

however little at may exceed $2l_1$.

Hence the impact terminates at the instant $t = 2l_1/a$.

Both bars are now free, the impinging bar (1) being free from compression throughout and having the uniform velocity V_2. The state of the longer bar (2) is given by

$$\left.\begin{aligned} \frac{\partial w_2}{\partial z_2} &= -F'(2l_1 - z_2) + F'(2l_1 + z_2 - 2l_2), \\ \frac{\partial w_2}{\partial t} &= a\,[F'(2l_1 - z_2) + F'(2l_1 + z_2 - 2l_2)] \end{aligned}\right\} \text{......(32)},$$

and $\qquad \partial w_2/\partial z_2 = 0$ when $z_2 = 0$ and when $z_2 = l_2$.

The form of the solution is different according as $2l_1$ is $<$ or $> l_2$.

(α) When $2l_1 < l_2$, measuring time from the instant of the termination of the impact, we may write

$$w_2 = \phi(at - z_2) + \phi(at + z_2 - 2l_2) \text{...........(33)};$$

and, since $\partial w_2/\partial z_2$ vanishes with z_2,

$$\phi'(\zeta) = \phi'(\zeta - 2l_2) \text{.....................(34)}.$$

The initial conditions (32) give us when $2l_1 > z_2 > 0$

$$-\phi'(-z_2) + \phi'(z_2 - 2l_2) = -\tfrac{1}{2}(V_1 - V_2)/a,$$
$$\phi'(-z_2) + \phi'(z_2 - 2l_2) = \tfrac{1}{2}(V_1 + V_2)/a;$$

and when $l_2 > z_2 > 2l_1$

$$-\phi'(-z_2) + \phi'(z_2 - 2l_2) = 0,$$
$$\phi'(-z_2) + \phi'(z_2 - 2l_2) = V_2/a.$$

$$\left.\begin{aligned} &\text{Hence when} \quad 0 > \zeta > -2l_1 \quad \phi'(\zeta) = \tfrac{1}{2}V_1/a, \\ &\text{and when} \quad -2l_1 > \zeta > -2l_2 \quad \phi'(\zeta) = \tfrac{1}{2}V_2/a \end{aligned}\right\} \text{.........(35)}.$$

The velocity of the end $z_2 = 0$ is $a\,[\phi'(at) + \phi'(at - 2l_2)]$ or $2a\phi'(at - 2l_2)$ and when $t = 0$ this is V_2. Thus, immediately after the impact terminates, this end moves with velocity V_2, and the ends of the two bars remain in contact with no pressure between them. The velocity remains unaltered until $at = 2(l_2 - l_1)$, when it

suddenly changes to V_1. Hence the bars separate after a time $2(l_2-l_1)/a$ from the instant when the impact ceases, or after a time $2l_2/a$ from the beginning of the impact.

(β) When $2l_1 > l_2$ we may write

$$w_2 = \psi(at - z_2) + \psi(at + z_2 - 2l_2) \qquad (36),$$

and

$$\psi'(\zeta) = \psi'(\zeta - 2l_2) \qquad (37);$$

and the initial conditions give us, when $2(l_2 - l_1) > z_2 > 0$,

$$-\psi'(-z_2) + \psi'(z_2 - 2l_2) = -\tfrac{1}{2}(V_1 - V_2)/a,$$

$$\psi'(-z_2) + \psi'(z_2 - 2l_2) = \tfrac{1}{2}(V_1 + V_2)/a,$$

and, when $l_2 > z_2 > 2(l_2 - l_1)$,

$$-\psi'(-z_2) + \psi'(z_2 - 2l_2) = 0,$$

$$\psi'(-z_2) + \psi'(z_2 - 2l_2) = V_1/a.$$

Hence when $0 > \zeta > -2l_1 \quad \psi'(\zeta) = \tfrac{1}{2}V_1/a,$
and when $-2l_1 > \zeta > -2l_2 \quad \psi'(\zeta) = \tfrac{1}{2}V_2/a$ ......(38).

The velocity of the end $z_2 = 0$ is as before $2a\psi'(at - 2l_2)$, and this is at first equal to V_2; so that the ends of the bars remain in contact without pressure. After a time $2(l_2 - l_1)/a$ from the instant when the ends become free the velocity changes to V_1, and the bars separate.

282. Physical Solution.

This problem can be solved synthetically as follows:—

When the two bars impinge the ends at the junction move with a common velocity $\frac{1}{2}(V_1 + V_2)$, and a compression $\frac{1}{2}(V_1 - V_2)/a$ is produced. Waves of compression run from the junction along both bars, and each element of either bar as the wave passes over it takes suddenly the velocity $\frac{1}{2}(V_1 + V_2)$ and the compression $\frac{1}{2}(V_1 - V_2)/a$. When the wave reaches the free end of the shorter bar it is reflected as a wave of extension; each element of the bar as the wave passes over it takes suddenly the velocity which initially belonged to the longer bar and zero extension. After a time equal to twice that required by a wave of compression to travel over the shorter bar, this bar has uniform velocity, equal to that which originally belonged to the longer bar, and no strain. The impact now ceases and there is in the longer bar a wave of compression of length equal to twice that of the shorter bar. The wave at this instant leaves the junction, and the junction end of

the longer bar takes a velocity equal to that which it had before the impact. The ends therefore remain in contact without pressure. This state of things continues until the wave returns reflected from the further end of the longer bar. When the time from the beginning of the impact is equal to twice that required by a wave of compression to travel over the longer bar, the junction end of the latter acquires suddenly a velocity equal to that originally possessed by the shorter bar, and the bars separate.

The shorter bar rebounds without strain and with the velocity of the longer, and the longer bar rebounds vibrating. Thus unless the bars be of equal length the theory shews that some of the energy will always be stored in the form of potential energy of strain in the longer bar, *i.e.* there is an apparent loss of kinetic energy. The amount of kinetic energy of the motions of the centres of inertia destroyed in the impact can be shewn to be

$$\tfrac{1}{2}\frac{m_1 m_2}{m_1+m_2}\left(1-\frac{m_1^2}{m_2^2}\right)(V_1-V_2)^2,$$

where m_1 is the mass of the shorter, and m_2 of the longer bar, while the relative velocity of the centres of inertia after impact has to that before impact the ratio $-m_1 : m_2$.

According to the ordinary theory of impact the ratio ought to be independent of the masses of the impinging bodies, and it is therefore clear that there is a discrepancy between the ordinary theory and that obtained by treating the bars as perfectly elastic and supposing them thrown into extensional vibrations by the impact.

283. Bars of different materials[1].

When the bars are of different materials, the problem can be solved in the same synthetic manner.

Let V_1, V_2 be the velocities, l_1, l_2 the lengths, and m_1, m_2 the masses of the bars, and let a_1, a_2 be the velocities of propagation of extensional waves along them. Suppose the bar (1) to impinge on the bar (2). The ends at the junction after impact move with the same velocity U and waves of compression travel along the bars. After a very short time τ lengths $a_1\tau$, $a_2\tau$ will be moving with the velocity U and the equation of constancy of momentum gives us

[1] Saint-Venant, *loc. cit.*

$$\left(\frac{m_1}{l_1}a_1\tau + \frac{m_2}{l_2}a_2\tau\right)U = \frac{m_1}{l_1}a_1\tau V_1 + \frac{m_2}{l_2}a_2\tau V_2,$$

or $$U = (V_1 + rV_2)/(1+r) \qquad \text{..................(39)},$$

where $$r = l_1a_2m_2/l_2a_1m_1 \qquad \text{.......................(40)}.$$

Each element of the bar (1) as the wave passes over it takes suddenly the velocity U and the compression $(V_1 - U)/a_1$, and each element of the bar (2) as the wave passes over it takes the velocity U and the compression $(U - V_2)/a_2$. When the wave reaches the free end of either bar it is reflected as a wave of extension. We shall suppose that the wave first reaches the free end of the bar (1) or that $l_1/a_1 < l_2/a_2$. Each element of the bar (1) as the reflected wave passes over it takes suddenly the velocity $2U - V_1$ and becomes free from compression or extension. After a time $2l_1/a_1$ from the beginning of the impact the reflected wave arrives at the junction and the bar (1) is unstrained and has velocity $V_2 - (V_1 - V_2)(r-1)/(r+1)$ throughout. At the same instant the junction end of the bar (2) becomes free and this end acquires velocity V_2. The bars separate, and the impact terminates, at this instant if $r > 1$, *i.e.* if $a_2m_2/l_2 > a_1m_1/l_1$. This condition expresses that the bar which the extensional waves traverse in the shorter time is also that in which the smaller mass is set in motion by a disturbance in a given time.

When this condition is fulfilled the bar (1) rebounds without strain and the bar (2) rebounds vibrating. The velocities of the centres of inertia of the bars (1) and (2) are

$$V_1 - \frac{2r}{r+1}(V_1 - V_2) \text{ and } V_2 + \frac{m_1}{m_2}\frac{2r}{r+1}(V_1 - V_2)$$

respectively, and the duration of impact is $2l_1/a_1$.

When the condition above specified is not fulfilled, but $r < 1$, the velocity of the junction end of the bar (1) immediately after the return of the reflected wave exceeds that of the junction end of the bar (2) and the impact recommences. A second wave is set up in each bar, and the formulæ are the same except that the velocity V_1 is to be replaced by $V_2 + (V_1 - V_2)(1-r)/(1+r)$, which was the velocity of the bar (1) immediately before the second impact. If the second wave returns reflected from the free end of the bar (1) before the first wave returns reflected from the free end of the bar (2), *i.e.* if $2l_1/a_1 < l_2/a_2$, the bar (1)

will at time $4l_1/a_1$ be unstrained and moving with velocity

$$V_2 + (V_1' - V_2)(1-r)/(1+r),$$

where $V_1' = V_2 + (V_1 - V_2)(1-r)/(1+r)$, *i.e.* the velocity of the bar (1) is $V_2 + (V_1 - V_2)(1-r)^2/(1+r)^2$. The same process takes place after every return of the wave from the free end of the bar (1), and if n be the greatest integer in l_2a_1/l_1a_2, so that $nl_1/a_1 < l_2/a_2 < (n+1)\,l_1/a_1$, the velocity of the junction end of the bar (1) will be reduced to $V_2 + (V_1 - V_2)(1-r)^n/(1+r)^n$ at the end of the time $2nl_1/a_1$.

Now at time t where $2l_2/a_2 > t > 2nl_1/a_1$ the velocity of the junction end of either bar is U_n, where

$$U_n(1+r) = V_2 + (V_1 - V_2)(1-r)^n/(1+r)^n + rV_2,$$

or

$$U_n = V_2 + (V_1 - V_2)(1-r)^n/(1+r)^{n+1} \;\ldots\ldots\; (41),$$

and the compression in the bar (1) at the junction end is

$$\frac{V_1 - V_2}{a_1}\, r \,\frac{(1-r)^n}{(1+r)^{n+1}} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (42),$$

while the compression in the bar (2) at the junction end is

$$\frac{V_1 - V_2}{a_2}\,\frac{(1-r)^n}{(1+r)^{n+1}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (43).$$

When $t = 2l_2/a_2$ the first wave in the bar (2) returns reflected from the free end. The compression of each element as the wave passes over it is diminished by $(U - V_2)/a_2$, so that at the instant $t = 2l_2/a_2$ the compression at the junction end of the bar (2) becomes

$$-\frac{V_1 - V_2}{a_2(1+r)}\left[1 - \frac{(1-r)^n}{(1+r)^n}\right] \;\ldots\ldots\ldots\ldots\ldots\; (44),$$

while the same end takes the velocity $U_n + (U - V_2)$ or

$$V_2 + \frac{V_1 - V_2}{1+r}\left[1 + \frac{(1-r)^n}{(1+r)^n}\right] \ldots\ldots\ldots\ldots\ldots (45).$$

The ends therefore become free and tend to separate. The velocity of the junction end of the bar (2) immediately after this instant is

$$V_2 + 2\,\frac{V_1 - V_2}{1+r},$$

which is found by subtracting the expression (44) multiplied by a_2 from the expression (45). The velocity of the junction end of the bar (1) at the same instant is

$$V_2 + (V_1 - V_2)(1-r)^n/(1+r)^n,$$

which is found by adding the expression (42) multiplied by a_1 to the expression (41).

Since $$2/(1+r) > (1-r)^n/(1+r)^n$$

we see that the bars separate.

The velocity of the centre of inertia of the bar (1) after separation is found by observing that a length $2l_2a_1/a_2 - 2nl_1$ [1] of it has velocity U_n given by (41), and the rest of it has velocity $V_2 + (V_1 - V_2)(1-r)^n/(1+r)^n$. The velocities of the centres of inertia are therefore for the bar (1)

$$v_1 = V_2 + (V_1 - V_2)\frac{(1-r)^n}{(1+r)^n}\left[1 - \frac{2r}{1+r}\left(\frac{a_1l_2}{a_2l_1} - n\right)\right] \ldots (46),$$

and for the bar (2)

$$v_2 = V_2 + \frac{m_1}{m_2}(V_1 - v_1) \ldots\ldots\ldots\ldots\ldots\ldots (47).$$

In this case both bars rebound vibrating and the duration of impact is $2l_2/a_2$.

We observe that by putting $r=1$ the results in this problem reduce to those in the case of like materials previously investigated. In all cases the duration of impact is twice the time taken by an extensional wave to travel over one of the bars. This does not accord with the results of Hamburger's experiments[2], according to which the duration of impact should be something like five times as great and should diminish slightly as the relative velocity before impact increases.

284. Theories of Impact.

The ordinary theory of impact founded by Newton divides bodies into two classes, "perfectly elastic" and "imperfectly elastic". In the impact of the former there is no loss of energy, while in the impact of the latter the amount of kinetic energy of the motions of the centres of inertia which disappears is the product of the harmonic mean of the masses, the square of the relative velocity before impact, and a coefficient depending on the materials of the impinging bodies.

[1] We have taken the case where this is $<l_1$, but the same results hold in the other case.

[2] Wiedemann's *Annalen*, XXVIII. 1886.

When two bodies impinge and mutually compress each other before separating it is clear that small relative motions will be set up in the parts near the surfaces that become common to the two bodies. Saint-Venant's theory takes account of these motions in the case of two bars impinging longitudinally, and it is a consequence of the theory that bodies which are in the ordinary sense perfectly elastic will not be so in the Newtonian sense. Other results of the theory are that the duration of impact is comparable with the gravest period of free vibration involving local compression at the impinging ends, and that the Newtonian "coefficient of restitution" depends upon the masses of the bodies.

Series of careful experiments on hard elastic bodies have been made with the view of deciding between the two theories. The results indicated wide differences from Saint-Venant's theory both as regards the coefficient of restitution and the duration of impact, while the velocities of the bodies after impact were found to be more nearly in accordance with the Newtonian theory of the impact of "perfectly elastic" bodies.

The following tables give some of the results of Prof. Voigt's[1] experiments on the longitudinal impact of bars of hard steel compared with the results of the Newtonian (perfectly elastic) and Saint-Venant's theories. The bars being of the same material and equal section and of lengths in the ratio 1 : 2, we denote by v_1 and v_2 the velocities of the centres of inertia of the shorter and the longer after impact. The first table gives the results when the longer impinges on the shorter at rest, the second when the shorter impinges on the longer at rest. The number in the first column gives the velocity of the impinging bar before impact.

TABLE I.

	Observed		Newtonian		Saint-Venant	
V_2	v_1	v_2	v_1	v_2	v_1	v_2
20	26·4	6·7	26·7	6·7	20	10
40	52·6	13·6	53·3	13·3	40	20
80·2	101·1	28·7	106·7	26·7	80·2	40·1

[1] Wiedemann's *Annalen*, xix. 1883. See also the account given by F. Auerbach in Winkelmann's *Handbuch der Physik* (Breslau, 1891). Bd. i. pp. 300, 301.

TABLE II.

V_1	Observed v_1	v_2	Newtonian v_1	v_2	Saint-Venant v_1	v_2
20	− 6·4	13·2	− 6·7	13·3	0	10
40	− 12·2	26·2	− 13·3	26·7	0	20
80·4	− 21·6	51·4	− 26·7	53·3	0	40·2

The explanation of the failure of Saint-Venant's theory must be sought in the supposition that it leaves out of account some essential circumstance. In the case of the "imperfectly elastic" bodies of the Newtonian theory, the consideration of the heat developed at the junction suggests that the energy apparently lost is really dissipated, *i.e.* converted into kinetic energy of molecular agitation, not into potential energy of strain and kinetic energy of relative molar motion of the parts of the impinging bodies. But in the cases where the amount of energy lost is actually very small, or the bodies are in the Newtonian sense almost perfectly elastic, it is more difficult to account for the discrepancy. The consideration of the heat developed will not help us because the vibrations of Saint-Venant's theory have already absorbed too much of the kinetic energy. What we want is a theory which without assuming dissipation of energy will explain why these vibrations do not take place.

285. Voigt's Theory.

Prof. Voigt[1] suggested that the source of error is to be found in Saint-Venant's terminal conditions at the junction. In Mechanics it is for some purposes sufficiently exact to treat bodies as continuous and bounded by surfaces, but there are some phenomena that can only be interpreted by means of the conception of bodies as congeries of molecules. In the theories of Capillarity and of Reflexion and Refraction of Polarized Light it is necessary to treat the parts of two media near their surface of separation as having different properties from the parts more remote. Instead of a separating surface we have to consider a

[1] Wiedemann's *Annalen*, XIX. 1883.

separating film, within which there is a rapid, but not sudden, transition from the properties of the first to those of the second medium. Prof. Voigt's theory of impact assumes the existence between two impinging bodies of a thin separating film, and his object is to attribute such properties to the film as in extreme conditions will include the Newtonian and Saint-Venant's theories as particular cases.

Consider the impact of two bars, and between the two at the junction let there be a separating film. Let l be the length of this film at the instant when the impact commences, and $l - \delta l$ its length at any time t during the continuance of the impact, then δl is the relative displacement towards each other of the ends of the two bars. The theory treats the film as a short massless elastic bar having a Young's modulus e, and a section ω. At time t this bar is uniformly compressed, and the pressure across any section of it is $e\omega\delta l/l$, and this is equal to the pressure on either end of the bar. The impact terminates when δl vanishes. If $e\omega/l$ be zero we have the Newtonian theory, and if $e\omega/l$ be infinite we have Saint-Venant's theory.

If z_1 and w_1 be the position and displacement at any section of the first bar, and z_2 and w_2 corresponding quantities for the second, the origins of z_1 and z_2 being at the junctions of the bars with the film, and the z's being measured towards the free ends, the terminal condition is that, when $z_1 = 0$ and $z_2 = 0$,

$$E_1\omega_1 \frac{\partial w_1}{\partial z_1} = \frac{e\omega}{l}(w_1 + w_2) = E_2\omega_2 \frac{\partial w_2}{\partial z_2},$$

where E_1, E_2 are the Young's moduluses, and ω_1, ω_2 the cross-sections of the two bars.

In some particular cases the solution has been worked out analytically by V. Hausmaninger. He found that the duration of the impact would be a little greater on Voigt's than on Saint-Venant's hypothesis, but when the constant e was adapted so as to make the results agree nearly with the Newtonian theory the duration of impact was still much less than that indicated by experiment. We shall not devote any more space to the description of this theory as it must be regarded as superseded by that which we shall next consider. The reader who wishes to

pursue the subject is referred to the memoirs of Voigt and Hausmaninger in Wiedemann's *Annalen*, XIX. and XXV.

286. Hertz's Theory.

We proceed to give an account of a quite different theory propounded by H. Hertz[1]. This may be described as an "equilibrium theory" inasmuch as it takes no account of vibrations set up in the bodies by the impact, but regards the compression at the junction as a local effect gradually produced and gradually subsiding. The theory is not adapted to the case of thin bars but to that of solids bounded by curved surfaces. In order that this theory may hold good it is necessary that the duration of the impact should be long compared with the gravest period of free vibration of either body which involves compression of the parts that come into contact. We shall see that as regards the motions of the centres of inertia of the two bodies the theory is in accord with Newton's, and in other respects it yields a satisfactory comparison with experiment.

Suppose that two bodies come into contact at certain points of them, and that the parts about these points are compressed, so that subsequently the contact is no longer confined to single points, but extends over a small finite area of the surface of each solid. Let us call this common surface the *compressed surface*, and the curve that bounds it at any time the *curve of pressure*, and let the resultant pressure between the two bodies across the compressed surface be P_0.

Let the surfaces of the two bodies in the neighbourhood of the point of contact, at the instant of the first contact, referred to the point of contact as origin be given by the equations

$$\left.\begin{aligned} z_1 &= A_1x^2 + B_1y^2 + 2Hxy, \\ z_2 &= A_2x^2 + B_2y^2 - 2Hxy \end{aligned}\right\} \quad \text{...............(48)},$$

where the axes of z_1 and z_2 are directed along the normals to the two bodies drawn towards the inside of each, and the axes of x and y have been so chosen as to make the H's the same. At the instant when the impact commences the distance between two corresponding points, one on each surface, which lie in the same

[1] 'Ueber die Beruhrung fester elastischer Körper'. Crelle-Borchardt, XCII. 1882.

plane through the common normal and at the same distance from it, is $(A_1 + A_2)x^2 + (B_1 + B_2)y^2$. In this expression the coefficients of x^2 and y^2 must have the same sign, and we choose the signs so as to make this expression positive.

We write
$$\left.\begin{aligned} A_1 + A_2 &= A, \\ B_1 + B_2 &= B \end{aligned}\right\} \quad \text{......................(49)},$$

and take α to be the relative displacement at any time of the two centres of inertia towards each other estimated in the direction of the common normal at the point of contact. Then the relative displacement of the two corresponding points of the surfaces which come into contact is

$$\alpha - Ax^2 - By^2.$$

Consider a system of fixed axes of x, y, z of which the axes of x and y are parallel to those to which the surfaces are referred. Let the plane $z = 0$ be the tangent plane at the point of contact at the instant when the impact commences, and let the axis z be directed towards the interior of the body (1). Also let u_1, v_1, w_1 be the displacements of any point of the body (1), and u_2, v_2, w_2 those of any point of the body (2), referred to these fixed axes; then we have, when $z = 0$, and x and y are very small,

$$w_1 - w_2 = \alpha - (Ax^2 + By^2) \quad \text{................(50)}.$$

The theory we are going to explain assumes:—

(a) That the problem is statical, or that the displacement at any time is that produced by the stress across the common surface at that time.

(b) That the common surface is always small and confined within a small closed curve, the *curve of pressure*, while the rest of the surface of each body is free.

(c) That within the curve of pressure the stress between the two bodies is in the direction of the common normal.

With these assumptions the problem reduces to solving the equations of elastic equilibrium for an infinite solid bounded by the plane $z = 0$ with the following conditions:—

(α) The displacements vanish at infinity.

(β) A small part of the bounding surface is subjected to purely normal pressure whose resultant is P_0, while the remainder is free from stress.

When this problem has been solved for each body we have the two conditions:—

(γ) The normal stresses are equal at all points within the curve of pressure.

(δ) Inside the curve of pressure corresponding points are brought together or

$$w_1 - w_2 = \alpha - Ax^2 - By^2,$$

and outside the curve of pressure the surfaces do not cross so that

$$w_1 - w_2 > \alpha - Ax^2 - By^2.$$

287. The Statical Problem.

The problem to be solved for each body (supposed isotropic) is a particular case of that which we have considered in I. ch. IX., arts. 160 sq. Assuming then that the bodies are isotropic, we shall have in the body (1)

$$\left.\begin{aligned} u_1 &= -\frac{1}{4\pi(\lambda_1+\mu_1)}\frac{\partial X}{\partial x} - \frac{1}{4\pi\mu_1}\frac{\partial^2\Phi}{\partial x\partial z}, \\ v_1 &= -\frac{1}{4\pi(\lambda_1+\mu_1)}\frac{\partial X}{\partial y} - \frac{1}{4\pi\mu_1}\frac{\partial^2\Phi}{\partial y\partial z}, \\ w_1 &= -\frac{1}{4\pi(\lambda_1+\mu_1)}\frac{\partial X}{\partial z} - \frac{1}{4\pi\mu_1}\frac{\partial^2\Phi}{\partial z^2} + \frac{\lambda_1+2\mu_1}{4\pi\mu_1(\lambda_1+\mu_1)}\nabla^2\Phi \end{aligned}\right\} \quad \text{...(51)},$$

where λ_1 and μ_1 are the elastic constants for the body (1); also we have

$$\left.\begin{aligned} X &= \iint \rho_1 \log(z+r)\, dx'dy', \\ \Phi &= \iint \rho_1 r\, dx'dy' \end{aligned}\right\} \quad \text{................(52)},$$

the integrations extending over the plane section of the compressed surface within the curve of pressure, and ρ_1 being the normal pressure per unit area.

Similar results hold for the body (2).

Now let $$P = \frac{1}{4\pi}\iint \frac{\rho_1}{r}\, dx'dy' \quad \text{......................(53)},$$

so that $4\pi P$ is the potential of a surface-density ρ_1 within the curve of pressure. We have

$$4\pi Pz = \frac{\partial\Phi}{\partial z}.$$

Let us write also

$$\Pi_1 = -\frac{zP}{\mu_1} - \frac{X}{4\pi(\lambda_1+\mu_1)} \quad \text{...............(54)};$$

then equations (51) may be replaced by

$$u_1 = \frac{\partial \Pi_1}{\partial x}, \quad v_1 = \frac{\partial \Pi_1}{\partial y}, \quad w_1 = \frac{\partial \Pi_1}{\partial z} + 2P\frac{\lambda_1 + 2\mu_1}{\mu_1(\lambda_1 + \mu_1)} \quad \text{...(55)},$$

which give the displacements of any point of the body (1) when the curve of pressure and the distribution of pressure on the compressed surface are known.

The resultant pressure P_0 between the two bodies is $\iint \rho_1 dx'dy'$ extended over the area within the curve of pressure, *i.e.* it is the quantity distributed with density ρ_1 to produce the potential $4\pi P$.

The displacement of any point on the surface (1) in the direction of the common normal is $P(\lambda_1 + 2\mu_1)/\{\mu_1(\lambda_1 + \mu_1)\}$, which we shall write $\vartheta_1 P$, so that $\vartheta_1 = 4(1 - \sigma_1^2)/E_1$, where E_1 is the Young's modulus and σ_1 the Poisson's ratio of the material of the body (1). In like manner that of the corresponding point on the surface (2) is $-\vartheta_2 P$ in the same direction, and therefore when $z = 0$ we have

$$\vartheta_2 w_1 + \vartheta_1 w_2 = 0 \quad \text{........................(56)}.$$

Hence, by (50), we have within the curve of pressure

$$\left.\begin{aligned} \vartheta_1 P = w_1 &= \frac{\vartheta_1}{\vartheta_1 + \vartheta_2}(\alpha - Ax^2 - By^2), \\ -\vartheta_2 P = w_2 &= -\frac{\vartheta_2}{\vartheta_1 + \vartheta_2}(\alpha - Ax^2 - By^2) \end{aligned}\right\} \quad \text{.........(57)},$$

and the equation of the compressed surface is $z = z_1 + w_1 = -(z_2 + w_2)$, or

$$(\vartheta_1 + \vartheta_2)z = -\vartheta_1 z_2 + \vartheta_2 z_1 \quad \text{...............(58)}.$$

288. The Curve of Pressure.

We shall now shew that if the curve of pressure be assumed to be an ellipse, and the density ρ_1 be taken to be the same as if there were an indefinitely flattened ellipsoid of mass P_0 occupying the interior of the curve of pressure, the dimensions of the ellipse can be determined so as to satisfy the second of the conditions (δ) as well as the conditions that

$$\iint \rho_1 dx'dy' = P_0, \quad \text{and} \quad \iint \frac{\rho_1}{r} dx'dy' = 4\pi P,$$

and that within the ellipse $(\vartheta_1 + \vartheta_2)P = (\alpha - Ax^2 - By^2)$ ...(59).

For suppose we have an ellipse of semiaxes a, b and distribute over it a mass P_0 so that the density at any point (x', y') is

$$Lt_{c=0}\left[\frac{3P_0}{4\pi abc}2c\sqrt{\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)}\right], \text{ or } \frac{3P_0}{2\pi ab}\sqrt{\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)};$$

its potential at any external point (x, y, z) will be $4\pi P$ provided

$$P=\frac{3P_0}{16\pi}\int_\nu^\infty\left(1-\frac{x^2}{a^2+\psi}-\frac{y^2}{b^2+\psi}-\frac{z^2}{\psi}\right)\frac{d\psi}{[(a^2+\psi)(b^2+\psi)\psi]^{\frac{1}{2}}}\text{...(60)},$$

where ν is the positive root of the equation

$$\frac{x^2}{a^2+\nu}+\frac{y^2}{b^2+\nu}+\frac{z^2}{\nu}=1 \text{ .... (61)};$$

and for the potential at an internal point the lower limit of the integral must be zero.

Thus we can satisfy equation (59) by taking

$$\left.\begin{aligned}\alpha&=(\vartheta_1+\vartheta_2)\frac{3P_0}{16\pi}\int_0^\infty\frac{d\psi}{[(a^2+\psi)(b^2+\psi)\psi]^{\frac{1}{2}}},\\ A&=(\vartheta_1+\vartheta_2)\frac{3P_0}{16\pi}\int_0^\infty\frac{d\psi}{(a^2+\psi)[(a^2+\psi)(b^2+\psi)\psi]^{\frac{1}{2}}},\\ B&=(\vartheta_1+\vartheta_2)\frac{3P_0}{16\pi}\int_0^\infty\frac{d\psi}{(b^2+\psi)[(a^2+\psi)(b^2+\psi)\psi]^{\frac{1}{2}}}\end{aligned}\right\}\text{...(62)}.$$

These are three equations to determine a, b, and P_0 in terms of A, B, and α. From the last two we can deduce an equation to determine the eccentricity e of the ellipse in terms of the ratio $A : B$. By writing $a^2\phi$ for ψ this equation becomes

$$B\int_0^\infty\frac{d\phi}{(1+\phi)^{\frac{3}{2}}[(1-e^2+\phi)\phi]^{\frac{1}{2}}}=A\int_0^\infty\frac{d\phi}{(1-e^2+\phi)^{\frac{3}{2}}[(1+\phi)\phi]^{\frac{1}{2}}}$$
$$\text{............(63)}.$$

Supposing e found from this equation, we have

$$\left.\begin{aligned}\alpha&=\frac{3P_0}{16\pi a}(\vartheta_1+\vartheta_2)\int_0^\infty\frac{d\phi}{[\phi(1+\phi)(1-e^2+\phi)]^{\frac{1}{2}}},\\ A&=\frac{3P_0}{16\pi a^3}(\vartheta_1+\vartheta_2)\int_0^\infty\frac{d\phi}{[\phi(1+\phi)^3(1-e^2+\phi)]^{\frac{1}{2}}}\end{aligned}\right\}\text{...(64)},$$

so that, on eliminating a, we find an equation of the form

$$\frac{\alpha^3}{A}=P_0^{\,2}(\vartheta_1+\vartheta_2)^2 f(e),$$

or

$$P_0=k_2\alpha^{\frac{3}{2}}\text{..........................(65)},$$

where k_2 depends only on the forms and materials of the two bodies.

The condition (δ) is that outside the curve of pressure the surfaces remain separate, or that outside the ellipse

$$w_1 - w_2 > (\alpha - Ax^2 - By^2).$$

This condition is that

$$(\vartheta_1 + \vartheta_2) P > \alpha - Ax^2 - By^2$$

when $(x^2/a^2 + y^2/b^2 - 1)$ is negative, and $z = 0$.

Now P is given by (60), which, when $z = 0$, may be written

$$(\vartheta_1 + \vartheta_2) P = (\alpha - Ax^2 - By^2) - \int_0^\nu \left(1 - \frac{x^2}{a^2+\psi} - \frac{y^2}{b^2+\psi}\right) \frac{d\psi}{[(a^2+\psi)(b^2+\psi)\psi]^{\frac{1}{2}}},$$

and when (x, y) lies on the ellipse (ν), and $\nu > \psi > 0$, the quantity $1 - x^2/(a^2+\psi) - y^2/(b^2+\psi)$ is constantly negative, so that this condition is satisfied.

289. Circumstances of the Impact.

The quantity α is the diminution up to time t in the projection on the common normal of the distance between the centres of inertia of the two bodies, so that $\dot{\alpha}$ is the relative velocity of the two centres of inertia parallel to this line. The pressure P_0 is equal to the rate of destruction of momentum of either body in the same direction, so that P_0 is proportional to $-\ddot{\alpha}$, and we may write

$$P_0 = -\ddot{\alpha}/k_1 \qquad\qquad (66),$$

where k_1 is a constant depending on the forms and masses of the two bodies.

Combining this equation and (65) we have

$$\ddot{\alpha} + k_1 k_2 \alpha^{\frac{3}{2}} = 0.$$

Multiplying by $\dot{\alpha}$ and integrating we have the equation

$$\dot{\alpha}^2 - \dot{\alpha}_0^2 + \tfrac{4}{5} k_1 k_2 \alpha^{\frac{5}{2}} = 0 \qquad\qquad (67),$$

where $\dot{\alpha}_0$ is the relative velocity of the centres of inertia in the direction of the common normal at the instant when the impact commences. This is really the equation of energy.

The greatest compression takes place when $\dot{\alpha}$ vanishes, and if α_1 be the value of α at this instant

$$\alpha_1 = \left[\frac{5\dot{\alpha}_0^2}{4k_1k_2}\right]^{\frac{2}{5}} \text{..........................(68).}$$

Before the instant of greatest compression the quantity α increases from zero to a maximum α_1, and $\dot{\alpha}$ diminishes from a maximum $\dot{\alpha}_0$ to zero. After the instant of greatest compression α diminishes from α_1 to zero and $\dot{\alpha}$ increases to $\dot{\alpha}_0$. The bodies then separate and the velocity with which they rebound is equal to that with which they approach. This result is in accord with Newton's Theory. It might have been predicted from the character of the fundamental assumptions.

The duration of the impact is

$$2\int_0^{\alpha_1} \frac{d\alpha}{\sqrt{(\dot{\alpha}_0^2 - \frac{4}{5}k_1k_2\alpha^{\frac{5}{2}})}},$$

and this is

$$2\frac{\alpha_1}{\dot{\alpha}_0}\int_0^1 \frac{dx}{(1-x^{\frac{5}{2}})^{\frac{1}{2}}}$$

$$= \frac{4}{5}\frac{\alpha_1}{\dot{\alpha}_0}\sqrt{\pi}\,\frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{9}{10})}$$

$$= \frac{\alpha_1}{\dot{\alpha}_0}(2\cdot9432) \text{ nearly..................(69),}$$

where α_1 is given by (68).

The duration of impact therefore varies inversely as the fifth root of the initial relative velocity.

The compressed surface at any time t is given by the equation

$$(\vartheta_1 + \vartheta_2)\,z = -\vartheta_1 z_2 + \vartheta_2 z_1,$$

where z_1 and z_2 are given by (48), and the curve of pressure is given by the equation

$$x^2/a^2 + y^2/b^2 = 1,$$

where a and b are given by (62).

290. Case of two spheres.

When two spheres impinge directly with relative velocity v, $\dot{\alpha}_0 = v$. Let m_1 and m_2 be the masses, and r_1 and r_2 the radii, then when the pressure is P_0

$$m_1\frac{d}{dt}\left(\frac{m_2}{m_1+m_2}\dot{\alpha}\right) = -P_0,$$

so that

$$k_1 = (m_1 + m_2)/m_1m_2.$$

We also have $$A_1 = B_1 = \frac{1}{2r_1},$$

$$A_2 = B_2 = \frac{1}{2r_2},$$

so that $$A = B = \tfrac{1}{2}(1/r_1 + 1/r_2).$$

Hence $a = b$ and the curve of pressure is a circle, while the compressed surface is part of a sphere of radius r such that

$$(\vartheta_1 + \vartheta_2)/r = -\vartheta_1/r_2 + \vartheta_2/r_1.$$

Also we have

$$\alpha = (\vartheta_1 + \vartheta_2)\frac{3P_0}{16\pi a}\int_0^\infty \frac{d\phi}{(1+\phi)\sqrt{\phi}} = \frac{3P_0}{16a}(\vartheta_1 + \vartheta_2),$$

$$\tfrac{1}{2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = (\vartheta_1 + \vartheta_2)\frac{3P_0}{16\pi a^3}\int_0^\infty \frac{d\phi}{(1+\phi)^2\sqrt{\phi}} = \frac{3P_0}{32a^3}(\vartheta_1 + \vartheta_2);$$

from which

$$k_2 = \tfrac{16}{3}\frac{1}{(\vartheta_1 + \vartheta_2)}\sqrt{\frac{r_1 r_2}{r_1 + r_2}}, \quad a^2 = \alpha\,(r_1 r_2)/(r_1 + r_2);$$

and therefore the duration of the impact is

$$(2{\cdot}9342)\left[\frac{225\,(\vartheta_1 + \vartheta_2)^2}{2^{12}\,v}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2\left(\frac{r_1 + r_2}{r_1 r_2}\right)\right]^{\frac{1}{5}} \quad \text{......(70).}$$

For the particular case of equal spheres of the same material the duration of the impact is

$$(2{\cdot}9432)\left[\frac{25\pi^2}{8}(1 - \sigma^2)\right]^{\frac{2}{5}}\frac{r}{v^{\frac{1}{5}}\,(E/\rho)^{\frac{2}{5}}} \quad \text{............(71).}$$

Experiments by Hertz, Schneebeli, and Hamburger on the impact of brass spheres and steel spheres and of spheres of ivory upon glass plates agreed very well with the theory. (See F. Auerbach in Winkelmann's *Handbuch der Physik*, pp. 304 and 305).

CHAPTER XVIII.

GENERAL THEORY OF WIRES NATURALLY CURVED.

291. Kirchhoff's Theory for wires naturally curved.

In art. 242 we have explained the elements of the theory of a rod or wire whose elastic central-line when unstrained is not straight, and which is such that, if it were simply unbent by turning each element through the angle of contingence in the osculating plane, and each osculating plane through the angle of torsion about the tangent, it would not be prismatic. Recalling the notation there employed, we suppose that in the unstrained state the component curvatures in the two principal planes are κ and λ, and that one principal axis of inertia (1) of the normal section through any point makes an angle ϕ_0 with the principal normal at the point[1]. Then if ρ and σ be the radii of curvature and tortuosity of the elastic central-line before strain, we have the rates of rotation of the principal torsion-flexure axes about themselves as we pass along the elastic central-line given by

$$\kappa = \sin\phi_0/\rho, \quad \lambda = \cos\phi_0/\rho, \quad \tau = d\phi_0/ds + 1/\sigma \text{......(1)}.$$

After strain we construct a system of moving rectangular axes of x, y, z whose origin is at a point P of the elastic central-line, whose z axis is the tangent to this line, and whose (z, x) plane contains the tangent to this line and the line-element that initially coincided with the principal axis (1) of the normal section.

Let P' be a neighbouring point on the elastic central-line, let ds be the unstrained length of PP', and ds' the length after strain, and let

$$ds' = ds\,(1+\epsilon) \text{........................(2)},$$

so that ϵ is the extension of the elastic central-line at P.

As we pass from P to P' the system of axes of x, y, z will execute a small rotation whose components about the axes of x, y, z

[1] The student reading this chapter for the first time is advised to work over arts. 292 to 296 with $\phi_0 = 0$.

at P are taken to be $\kappa' ds$, $\lambda' ds$, $\tau' ds$; then $\kappa' - \kappa$, $\lambda' - \lambda$, are the component changes of curvature, and $\tau' - \tau$ measures the twist.

We have seen that the stress-couples at any section are $A(\kappa' - \kappa)$, $B(\lambda' - \lambda)$, $C(\tau' - \tau)$, where A and B are the principal flexural rigidities, and C is the torsional rigidity.

As there is some controversy[1] about this result it may be as well to indicate another method of proof. We may adapt the theory of arts. 250 to 257 to our purpose. It is not difficult to shew[2] that the relative displacements of points within an elementary prism are connected by equations of the same form as (7) of art. 251 with $\kappa' - \kappa$, $\lambda' - \lambda$, $\tau' - \tau$ written in place of κ, λ, τ. We shall have the same classification of cases and method of approximation as in art. 252, and we can deduce similar results. We may conclude that whenever there is flexure the stress-couples are given by the forms stated, but the stress-resultants are unknown, and that, when the elastic central-line is simply extended, there is a tension T equal to the product of ϵ, the Young's modulus, and the area of the normal section, and the remaining stress-resultants and the stress-couples are unimportant.

292. Infinitely small displacements[3].

We shall now suppose a wire which in the natural state is finitely curved to be very slightly deformed. It is supposed that the wire is of uniform section, and before strain its elastic central-line forms a tortuous curve of curvature $1/\rho$ and tortuosity $1/\sigma$. The principal axes of inertia of a normal section at its centroid make with the principal normal and binormal before strain angles equal to ϕ_0, some function of the arc s measured from a fixed point on the elastic central-line. The deformation is such that the centroid of each normal section moves through distances w, u, v parallel to the tangent to the elastic central-line and to the principal axes of inertia of the normal section in the unstrained

[1] See Basset on the 'Theory of Elastic Wires'. *Proc. Lond. Math. Soc.* XXIII. 1892.

[2] We shall be obliged to undertake a precisely similar piece of work later in connexion with the theory of thin elastic shells, (see below, ch. XXI.) and it is not necessary to give the corresponding investigation here.

[3] Cf. Michell, *Messenger of Mathematics*, XIX. 1890. The investigations there given are less general than those of this chapter inasmuch as the section is taken to be circular, and an assumption is made which is equivalent to supposing that the same set of transverses of the wire which are initially principal normals continue to be principal normals after strain (cf. art. 242).

state, while the line-elements initially coinciding with these axes execute small rotations. We shall consider in the first place the deformation of the elastic central-line.

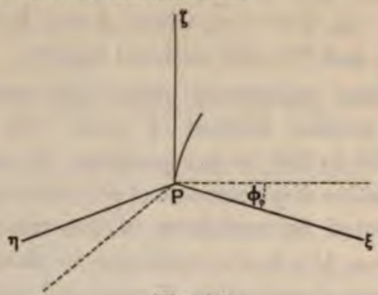


Fig. 44.

Let P be a point on the elastic central-line, and draw from the unstrained position of P a system of fixed axes of ξ, η, ζ which coincide with the unstrained position of the three lines of reference above mentioned. In figure 44 the dotted lines shew the principal normal and binormal of the elastic central-line at P. The axis ξ makes an angle ϕ_0 with the principal normal, and the axis η an angle ϕ_0 with the binormal. If we imagine a system of axes to move along the unstrained elastic central-line so as to be at every point identical with the principal axes of inertia of the normal section and the tangent to the elastic central-line, these axes will coincide with the axes of ξ, η, ζ at P, and their positions at a neighbouring point P' of the elastic central-line, distant ds from P, will be obtained from those at P by a translation ds along the axis ζ, and rotations $\sin\phi_0\, ds/\rho$, $\cos\phi_0\, ds/\rho$, $d\phi_0 + ds/\sigma$ about the axes ξ, η, ζ. The system of moving axes thus obtained is the system of lines of reference for the displacements u, v, w.

Let ξ, η, ζ be the coordinates of P after strain and $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$ those of the neighbouring point P'. Then ξ, η, ζ are identical with u, v, w, but $d\xi, d\eta, d\zeta$ are not identical with du, dv, dw, since u, v, w are referred to moving axes. In fact we have

$$\left.\begin{aligned} d\xi &= d\mathrm{u} - \mathrm{v}\left(d\phi_0 + \frac{ds}{\sigma}\right) + \mathrm{w}\cos\phi_0\frac{ds}{\rho}, \\ d\eta &= d\mathrm{v} - \mathrm{w}\sin\phi_0\frac{ds}{\rho} + \mathrm{u}\left(d\phi_0 + \frac{ds}{\sigma}\right), \\ d\zeta &= d\mathrm{w} - \mathrm{u}\cos\phi_0\frac{ds}{\rho} + \mathrm{v}\sin\phi_0\frac{ds}{\rho} + ds \end{aligned}\right\} \qquad (3).$$

If ds' be the stretched length of PP' we find, by squaring and adding and rejecting squares and products of u, v, w and their differential coefficients,

$$(ds')^2 = ds^2 \left\{1 + \frac{dw}{ds} - u\frac{\cos\phi_0}{\rho} + v\frac{\sin\phi_0}{\rho}\right\}^2 \;\ldots\ldots\; (4).$$

If ϵ be the extension of the elastic central-line, so that

$$ds' = ds\,(1+\epsilon),$$

we shall have
$$\epsilon = \frac{dw}{ds} - u\frac{\cos\phi_0}{\rho} + v\frac{\sin\phi_0}{\rho} \;\ldots\ldots\ldots\ldots\; (5),$$

and the condition of inextensibility is

$$\frac{dw}{ds} - u\frac{\cos\phi_0}{\rho} + v\frac{\sin\phi_0}{\rho} = 0 \;\ldots\ldots\ldots\ldots\; (6).$$

293. Curvature and Tortuosity.

Let λ_3, μ_3, ν_3 be the direction-cosines of the elastic central-line PP' after strain referred to the lines of reference of u, v, w, and l_3, m_3, n_3 the direction-cosines of the same line referred to the axes of ξ, η, ζ. Then l_3, m_3, n_3 are identical with λ_3, μ_3, ν_3, but dl_3, dm_3, dn_3 are not identical with $d\lambda_3$, $d\mu_3$, $d\nu_3$; in fact

$$\left.\begin{aligned} dl_3 &= d\lambda_3 - \mu_3\left(d\phi_0 + \frac{ds}{\sigma}\right) + \nu_3\cos\phi_0\frac{ds}{\rho}, \\ dm_3 &= d\mu_3 - \nu_3\sin\phi_0\frac{ds}{\rho} + \lambda_3\left(d\phi_0 + \frac{ds}{\sigma}\right), \\ dn_3 &= d\nu_3 - \lambda_3\cos\phi_0\frac{ds}{\rho} + \mu_3\sin\phi_0\frac{ds}{\rho} \end{aligned}\right\}\ldots\ldots\ldots\; (7).$$

Now $\quad \lambda_3 = l_3 = d\xi/ds',\ \mu_3 = m_3 = d\eta/ds',\ \nu_3 = n_3 = d\zeta/ds' \ldots (8)$,

or, neglecting squares and products of u, v, w and their differential coefficients, we have

$$\left.\begin{aligned} \lambda_3 &= \frac{du}{ds} - v\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + w\frac{\cos\phi_0}{\rho}, \\ \mu_3 &= \frac{dv}{ds} - w\frac{\sin\phi_0}{\rho} + u\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right), \\ \nu_3 &= 1 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\; (9).$$

From these dl_3, dm_3, dn_3 are easily written down by (7).

The angle of contingence of the elastic central-line after strain

is $\{(dl_3)^2+(dm_3)^2+(dn_3)^2\}^{\frac{1}{2}}$; and, rejecting squares of u, v, w..., we may omit $(dn_3)^2$. We thus get for the angle of contingence

$$ds\left\{\left[\frac{\cos\phi_0}{\rho}+\frac{d}{ds}\left\{\frac{du}{ds}-v\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)+w\,\frac{\cos\phi_0}{\rho}\right\}\right.\right.$$
$$\left.-\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\left\{\frac{dv}{ds}-w\,\frac{\sin\phi_0}{\rho}+u\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\right\}\right]^2$$
$$+\left[-\frac{\sin\phi_0}{\rho}+\frac{d}{ds}\left\{\frac{dv}{ds}-w\,\frac{\sin\phi_0}{\rho}+u\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\right\}\right.$$
$$\left.\left.+\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\left\{\frac{du}{ds}-v\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)+w\,\frac{\cos\phi_0}{\rho}\right\}\right]^2\right\}^{\frac{1}{2}}\ldots(10).$$

From this we easily find the curvature of the elastic central-line after strain by dividing by ds' or

$$ds\left\{1+\frac{dw}{ds}-u\,\frac{\cos\phi_0}{\rho}+v\,\frac{\sin\phi_0}{\rho}\right\}.$$

If ρ' be the radius of curvature, we have, rejecting terms of the second order in u, v, w,

$$\frac{1}{\rho'}-\frac{1}{\rho}=\frac{1}{\rho}\left\{-\frac{dw}{ds}+u\,\frac{\cos\phi_0}{\rho}-v\,\frac{\sin\phi_0}{\rho}\right\}$$
$$+\cos\phi_0\left[\frac{d}{ds}\left\{\frac{du}{ds}-v\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)+w\,\frac{\cos\phi_0}{\rho}\right\}\right.$$
$$\left.-\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\left\{\frac{dv}{ds}-w\,\frac{\sin\phi_0}{\rho}+u\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\right\}\right]$$
$$-\sin\phi_0\left[\frac{d}{ds}\left\{\frac{dv}{ds}-w\,\frac{\sin\phi_0}{\rho}+u\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\right\}\right.$$
$$\left.+\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)\left\{\frac{du}{ds}-v\left(\frac{d\phi_0}{ds}+\frac{1}{\sigma}\right)+w\,\frac{\cos\phi_0}{\rho}\right\}\right]\ldots(11),$$

where the first line vanishes if the central-line be unextended.

Let λ', μ', ν' be the direction-cosines of the binormal of the elastic central-line after strain referred to the lines of reference of u, v, w, and l', m', n' the direction-cosines of the same line referred to the axes of ξ, η, ζ. Then as before l', m', n' are identical with λ', μ', ν', but

$$dl'=d\lambda'-\mu'\left(d\phi_0+\frac{ds}{\sigma}\right)+\nu'\cos\phi_0\frac{ds}{\rho}\ \ldots\ldots\ (12),$$

and dm' and dn' are given by similar equations.

Now $$\frac{\lambda'}{m_3dn_3-n_3dm_3}=\frac{\mu'}{n_3dl_3-l_3dn_3}=\frac{\nu'}{l_3dm_3-m_3dl_3}.$$

Observing that l_3, m_3, and dn_3 contain no terms independent of u, v, w, we can simplify these to

$$\frac{\lambda'}{-dm_3} = \frac{\mu'}{dl_3} = \frac{\nu'}{l_3 dm_3 - m_3 dl_3} \quad\text{..............(13).}$$

Again, $$dl_3 = \left[\frac{\cos\phi_0}{\rho} + \frac{d\lambda_3}{ds} - \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\mu_3\right]ds,$$

and $$dm_3 = \left[-\frac{\sin\phi_0}{\rho} + \frac{d\mu_3}{ds} + \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\lambda_3\right]ds;$$

so that

$$l_3 dm_3 - m_3 dl_3 = \left[-\frac{\sin\phi_0}{\rho}\lambda_3 - \frac{\cos\phi_0}{\rho}\mu_3\right]ds.$$

Hence the square root of the sum of the squares of the denominators of (13) is ultimately the square root of $(dl_3)^2 + (dm_3)^2$ and this has been already found.

Rejecting squares of u, v, w we find

$$\left.\begin{aligned}
\lambda' = \sin\phi_0 - \rho\cos^2\phi_0 &\left[\frac{d}{ds}\left\{\frac{d\mathrm{v}}{ds} - \mathrm{w}\frac{\sin\phi_0}{\rho} + \mathrm{u}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}\right. \\
&\left. + \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\}\right] \\
- \rho\sin\phi_0\cos\phi_0 &\left[\frac{d}{ds}\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\}\right. \\
&\left. - \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\left\{\frac{d\mathrm{v}}{ds} - \mathrm{w}\frac{\sin\phi_0}{\rho} + \mathrm{u}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}\right], \\
\mu' = \cos\phi_0 + \rho\sin^2\phi_0 &\left[\frac{d}{ds}\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\}\right. \\
&\left. - \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\left\{\frac{d\mathrm{v}}{ds} - \mathrm{w}\frac{\sin\phi_0}{\rho} + \mathrm{u}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}\right] \\
+ \rho\sin\phi_0\cos\phi_0 &\left[\frac{d}{ds}\left\{\frac{d\mathrm{v}}{ds} - \mathrm{w}\frac{\sin\phi_0}{\rho} + \mathrm{u}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}\right. \\
&\left. + \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\}\right], \\
\nu' = -\sin\phi_0 &\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\} \\
- \cos\phi_0 &\left\{\frac{d\mathrm{v}}{ds} - \mathrm{w}\frac{\sin\phi_0}{\rho} + \mathrm{u}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}
\end{aligned}\right\} \quad\text{...........(14).}$$

With these values of λ', μ', ν' we have further

$$\left.\begin{aligned} dl' &= d\lambda' - \mu'\left(d\phi_0 + \frac{ds}{\sigma}\right) + \nu' \cos\phi_0 \frac{ds}{\rho}, \\ dm' &= d\mu' - \nu' \sin\phi_0 \frac{ds}{\rho} + \lambda'\left(d\phi_0 + \frac{ds}{\sigma}\right), \\ dn' &= d\nu' - \lambda' \cos\phi_0 \frac{ds}{\rho} + \mu' \sin\phi_0 \frac{ds}{\rho} \end{aligned}\right\} \qquad \text{(15)},$$

and the measure of tortuosity $1/\sigma'$ of the strained elastic central-line is given by

$$1/\sigma' = \{(dl')^2 + (dm')^2 + (dn')^2\}^{\frac{1}{2}}/ds' \qquad \text{(16)}.$$

The principal normal of the strained elastic central-line has direction-cosines proportional to dl_3, dm_3, and dn_3, and the square root of the sum of the squares has been already found. We get therefore for the direction-cosines l, m, n of the principal normal

$$\left.\begin{aligned} & l = \mu', \qquad m = -\lambda', \\ & n = -\cos\phi_0 \left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\} \\ & \qquad + \sin\phi_0 \left\{\frac{d\mathrm{v}}{ds} - \mathrm{w}\frac{\sin\phi_0}{\rho} + \mathrm{u}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\} \end{aligned}\right\} \qquad \text{(17)},$$

where μ' and λ' are given by the first two of (14).

294. Simplified forms[1].

All the expressions that we have obtained become very much simplified when ϕ_0 is identically zero. This will happen if, in the natural state of the wire, either the sections are circular, or one principal axis of inertia of each normal section lies in the osculating plane of the elastic central-line.

For this case we find the following:

The extension of the elastic central-line is

$$\frac{d\mathrm{w}}{ds} - \frac{\mathrm{u}}{\rho} \qquad \text{(18)}.$$

The direction-cosines of the tangent are

$$\frac{d\mathrm{u}}{ds} - \frac{\mathrm{v}}{\sigma} + \frac{\mathrm{w}}{\rho}, \quad \frac{d\mathrm{v}}{ds} + \frac{\mathrm{u}}{\sigma}, \quad 1 \qquad \text{(19)}.$$

[1] Some of the results have been given by Michell and Basset.

The curvature of the elastic central-line after strain is $1/\rho'$, where

$$\frac{1}{\rho'}-\frac{1}{\rho}=\frac{d^2\mathrm{u}}{ds^2}+\left(\frac{1}{\rho^2}-\frac{1}{\sigma^2}\right)\mathrm{u}-\frac{2}{\sigma}\frac{d\mathrm{v}}{ds}-\mathrm{v}\frac{d}{ds}\left(\frac{1}{\sigma}\right)+\mathrm{w}\frac{d}{ds}\left(\frac{1}{\rho}\right)\text{......} \quad (20).$$

The direction-cosines of the principal normal to the strained elastic central-line are l, m, n, where

$$\left.\begin{aligned} l&=1,\\ m&=\rho\left\{\frac{d}{ds}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)+\frac{1}{\sigma}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\right\},\\ n&=-\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\end{aligned}\right\}\text{.........}(21).$$

The direction-cosines of the binormal are l', m', n', where

$$\left.\begin{aligned} l'&=-\rho\left\{\frac{d}{ds}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)+\frac{1}{\sigma}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\right\},\\ m'&=\ 1,\\ n'&=-\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)\end{aligned}\right\}\text{.........}(22).$$

To find the measure of tortuosity $1/\sigma'$ we have

$$dl'=\left[-\frac{d}{ds}\left\{\rho\frac{d}{ds}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)+\frac{\rho}{\sigma}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\right\}-\frac{1}{\sigma}-\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)\frac{1}{\rho}\right]ds,$$

while all the terms in dm' and dn' are of the same order as u, v, w. Hence

$$\frac{1}{\sigma'}=\left(1-\frac{d\mathrm{w}}{ds}+\frac{\mathrm{u}}{\rho}\right)\left[\frac{1}{\sigma}+\frac{d}{ds}\left\{\rho\frac{d}{ds}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)+\frac{\rho}{\sigma}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\right\}\right.$$
$$\left.+\frac{1}{\rho}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)\right],$$

or

$$\frac{1}{\sigma'}-\frac{1}{\sigma}=\frac{d}{ds}\left\{\rho\frac{d}{ds}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)+\frac{\rho}{\sigma}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\right\}$$
$$+\frac{1}{\rho}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)-\frac{1}{\sigma}\left(\frac{d\mathrm{w}}{ds}-\frac{\mathrm{u}}{\rho}\right)\text{......}(23),$$

where the last term vanishes if the central-line be unextended.

295. The lines of reference.

For the purpose of forming the equations of elastic equilibrium or small motion of the wire we have to investigate expressions for the quantities κ', λ', τ' introduced and defined in art. 242. A system of moving rectangular axes of x, y, z is to be constructed, with its origin at a point P of the strained elastic central-line,

the axis z coinciding with the tangent to this line after strain, and the plane (x, z) containing the line-element which before strain coincided with that principal axis of inertia of the normal section at P which was inclined at an angle ϕ_0 to the principal normal.

We shall suppose as in art. 232 that, referred to the fixed axes of ξ, η, ζ at P, these axes of x, y, z are given by a scheme of 9 direction-cosines

$$\left.\begin{array}{cccc} & \xi & \eta & \zeta \\ x & l_1, & m_1, & n_1 \\ y & l_2, & m_2, & n_2 \\ z & l_3, & m_3, & n_3 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(24)},$$

and we shall further suppose that, referred to the lines of reference for u, v, w at P (art. 292), the same axes are given by a scheme of 9 direction-cosines

$$\left.\begin{array}{cccc} & \mathrm{u} & \mathrm{v} & \mathrm{w} \\ x & \lambda_1, & \mu_1, & \nu_1 \\ y & \lambda_2, & \mu_2, & \nu_2 \\ z & \lambda_3, & \mu_3, & \nu_3 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(25)}.$$

Then *e.g.* l_1, m_1, n_1, are identical with λ_1, μ_1, ν_1 but

$$dl_1 = d\lambda_1 - \mu_1\left(d\phi_0 + \frac{ds}{\sigma}\right) + \nu_1 \cos\phi_0 \frac{ds}{\rho} \quad \ldots\ldots\ldots \text{(26)},$$

and the differentials of the others are given by similar equations. In particular l_3, m_3, n_3 and their differentials have already been found in art. 293.

After strain the line-element (1) which initially coincided with the axis ξ will make an angle $\frac{1}{2}\pi - \gamma$ with the axis (3), where γ is indefinitely small, and the plane through the line-elements (1) and (3) which initially coincided with the plane (ξ, ζ) will make an indefinitely small angle β with the initial position of the axis (1). The direction-cosines of the strained position of the line-element (1) are ultimately $1, \beta, \gamma$, and we can find the direction-cosines of the axes x, and y, in terms of those of the axis z and the angle β.

The condition that the lines $(\lambda_1, \mu_1, \nu_1)$, $(1, \beta, \gamma)$, $(\lambda_3, \mu_3, \nu_3)$ lie in one plane is

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ 1 & \beta & \gamma \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = 0,$$

and the condition that $(\lambda_1, \mu_1, \nu_1)$ and $(\lambda_3, \mu_3, \nu_3)$ are at right angles is

$$\lambda_1\lambda_3 + \mu_1\mu_3 + \nu_1\nu_3 = 0.$$

Solving these, and remembering that β, γ, and λ_3, μ_3 are very small, while ν_3 is ultimately unity, we find

$$\lambda_1 = 1, \quad \mu_1 = \beta, \quad \nu_1 = -\lambda_3 \quad \text{..................(27).}$$

Since $(\lambda_2, \mu_2, \nu_2)$ is at right angles to $(\lambda_1, \mu_1, \nu_1)$ and to $(\lambda_3, \mu_3, \nu_3)$, we have

$$\lambda_2 = -\beta, \quad \mu_2 = 1, \quad \nu_2 = -\mu_3 \quad \text{...............(28).}$$

In these λ_3 and μ_3 have the values given in equations (9) of art. 293.

296. Component curvatures and twist.

We are now in a position to calculate κ', λ', τ', where $\kappa' ds'$, $\lambda' ds'$, $\tau' ds'$ are the infinitesimal rotations executed by the axes of x, y, z about themselves in passing from P to P'. As in art. 232 we have

$$\left.\begin{aligned} \kappa' ds' &= l_3 dl_2 + m_3 dm_2 + n_3 dn_2 = -\{l_2 dl_3 + m_2 dm_3 + n_2 dn_3\}, \\ \lambda' ds' &= l_1 dl_3 + m_1 dm_3 + n_1 dn_3, \\ \tau' ds' &= l_2 dl_1 + m_2 dm_1 + n_2 dn_1 \end{aligned}\right\} \text{...(29).}$$

As we have already found dl_3, dm_3, dn_3 we only require further to find dl_1, dm_1, dn_1.

Now
$$dl_1 = d\lambda_1 - \mu_1\left(d\phi_0 + \frac{ds}{\sigma}\right) + \nu_1 \cos\phi_0 \frac{ds}{\rho},$$

$$dm_1 = d\mu_1 - \nu_1 \sin\phi_0 \frac{ds}{\rho} \quad + \lambda_1\left(d\phi_0 + \frac{ds}{\sigma}\right),$$

$$dn_1 = d\nu_1 - \lambda_1 \cos\phi_0 \frac{ds}{\rho} \quad + \mu_1 \sin\phi_0 \frac{ds}{\rho}.$$

Hence

$$\left.\begin{aligned} dl_1 &= -\beta\left(d\phi_0 + \frac{ds}{\sigma}\right) - \cos\phi_0 \frac{ds}{\rho}\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\}, \\ dm_1 &= d\beta + \sin\phi_0 \frac{ds}{\rho}\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\} + d\phi_0 + \frac{ds}{\sigma}, \\ dn_1 &= -d\left\{\frac{d\mathrm{u}}{ds} - \mathrm{v}\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \mathrm{w}\frac{\cos\phi_0}{\rho}\right\} - \cos\phi_0 \frac{ds}{\rho} + \sin\phi_0 \frac{ds}{\rho}\beta \end{aligned}\right\}$$
$$\text{..................(30).}$$

We therefore have, rejecting squares of β, u, v, w...,

$$\left.\begin{aligned}
\kappa' &= \beta\frac{\cos\phi_0}{\rho} + \frac{(1-\epsilon)\sin\phi_0}{\rho} - \frac{d}{ds}\left\{\frac{dv}{ds} - w\frac{\sin\phi_0}{\rho} + u\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\} \\
&\qquad - \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\left\{\frac{du}{ds} - v\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + w\frac{\cos\phi_0}{\rho}\right\}, \\
\lambda' &= -\beta\frac{\sin\phi_0}{\rho} + \frac{(1-\epsilon)\cos\phi_0}{\rho} + \frac{d}{ds}\left\{\frac{du}{ds} - v\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + w\frac{\cos\phi_0}{\rho}\right\} \\
&\qquad - \left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\left\{\frac{dv}{ds} - w\frac{\sin\phi_0}{\rho} + u\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}, \\
\tau' &= \frac{d\beta}{ds} + (1-\epsilon)\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + \frac{\sin\phi_0}{\rho}\left\{\frac{du}{ds} - v\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + w\frac{\cos\phi_0}{\rho}\right\} \\
&\qquad + \frac{\cos\phi_0}{\rho}\left\{\frac{dv}{ds} - w\frac{\sin\phi_0}{\rho} + u\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right)\right\}
\end{aligned}\right\} \quad \text{..................(31)},$$

where ϵ is given by (5) of art. 292.

We could hence find the new measure of tortuosity $1/\sigma'$ by using the equation $\tau' = \frac{1}{\sigma'} + \frac{d}{ds'}\left(\tan^{-1}\frac{\kappa'}{\lambda'}\right)$. This would give us the same value as that found in art. 294 when $\phi_0 = 0$. We could find the principal curvature $1/\rho'$ from the equation

$$(1/\rho')^2 = \kappa'^2 + \lambda'^2$$

and this will give us the same value as that found in art. 293.

In the notation of art. 291 the values of κ, λ, τ can be deduced from those of κ', λ', τ' by putting u, v, w, β, and ϵ all zero and we find

$$\kappa = \frac{\sin\phi_0}{\rho}, \quad \lambda = \frac{\cos\phi_0}{\rho}, \quad \tau = \frac{d\phi_0}{ds} + \frac{1}{\sigma},$$

so that, as already stated, these are the rates of rotation (per unit of length of the arc) of the lines of reference for u, v, w about themselves.

297. Equations of equilibrium.

We have already investigated certain modes of finite deformation of the wire, and for these $\kappa' - \kappa$, $\lambda' - \lambda$, $\tau' - \tau$ are finite while ϵ is infinitesimal. There will exist modes of infinitely small deformation continuous with these for which $\epsilon \div$ the unit of length is infinitely small in comparison with $\kappa' - \kappa$, $\lambda' - \lambda$, $\tau' - \tau$. In these modes the elastic central-line remains unextended, and we

shall now investigate the equations of equilibrium on this supposition.

The equations of equilibrium of an element of the wire contained between two consecutive normal sections, when the wire is subject at every point to forces X, Y, Z and couples L, M, N per unit length, along and about axes coinciding with the two principal axes of inertia of a normal section and the tangent to the elastic central-line, can be written down in terms of the shearing forces N_1, N_2, and the tension T, and the flexural couples G_1, G_2, and the torsional couple H arising from the elastic reactions. The equations take the forms

$$\left.\begin{aligned}
&\frac{dN_1}{ds} - N_2\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + T\frac{\cos\phi_0}{\rho} + X = 0,\\
&\frac{dN_2}{ds} - T\frac{\sin\phi_0}{\rho} + N_1\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + Y = 0,\\
&\frac{dT}{ds} - N_1\frac{\cos\phi_0}{\rho} + N_2\frac{\sin\phi_0}{\rho} + Z = 0
\end{aligned}\right\}\text{.......(32)},$$

and

$$\left.\begin{aligned}
&\frac{dG_1}{ds} - G_2\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + H\frac{\cos\phi_0}{\rho} - N_2 + L = 0,\\
&\frac{dG_2}{ds} - H\frac{\sin\phi_0}{\rho} + G_1\left(\frac{d\phi_0}{ds} + \frac{1}{\sigma}\right) + N_1 + M = 0,\\
&\frac{dH}{ds} - G_1\frac{\cos\phi_0}{\rho} + G_2\frac{\sin\phi_0}{\rho} + N = 0
\end{aligned}\right\}\text{.....(33)}.$$

In these equations

$$\left.\begin{aligned}
G_1 &= A\left(\kappa' - \frac{\sin\phi_0}{\rho}\right),\\
G_2 &= B\left(\lambda' - \frac{\cos\phi_0}{\rho}\right),\\
H &= C\left(\tau' - \frac{d\phi_0}{ds} - \frac{1}{\sigma}\right)
\end{aligned}\right\}\text{..................(34)},$$

where κ', λ', τ' are given by equations (31) in which ϵ is put equal to zero.

The above equations with the condition of inextensibility (5) viz.

$$\frac{dw}{ds} - u\frac{\cos\phi_0}{\rho} + v\frac{\sin\phi_0}{\rho} = 0$$

constitute the general equations of equilibrium of the wire.

The class of cases here considered, viz. those in which the elastic central-line remains unextended, includes almost all that are of any importance. Just as in art. 257 we can see that either this supposition (of no extension) is legitimate, or else in general the extension is the important thing. For example, in the case of a wire initially circular and strained into a new circle by uniform normal pressure, the extension gives rise to a tension which is of a much lower order of small quantities than the couple that arises from the change of curvature. Such cases are exceptional. More generally the reverse is the case, and the couples arising from change of curvature, although containing quantities of the order of the fourth power of the linear dimensions of the cross-section, are nevertheless of a lower order of small quantities than the part of the tension which is proportional to the extension of the elastic central-line and the area of the normal section. It is easy to see that the condition of inextensibility and the equations of equilibrium in the form given in this article in general lead to as many equations as there are quantities to be determined, while if the condition of inextensibility be omitted and the couples be rejected as small, we shall have three equations containing T and the forces, which will be of the same form as the equations of equilibrium of an elastic string.

298. Simplified forms of the equations.

In the case where $\phi_0 = 0$, explained in art. 294, there is a considerable simplification; we have in fact

$$\left.\begin{aligned} \kappa' &= \frac{\beta}{\rho} - \frac{d}{ds}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{du}{ds} - \frac{v}{\sigma} + \frac{w}{\rho}\right), \\ \lambda' &= \frac{1}{\rho} + \frac{d}{ds}\left(\frac{du}{ds} - \frac{v}{\sigma} + \frac{w}{\rho}\right) - \frac{1}{\sigma}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right), \\ \tau' &= \frac{d\beta}{ds} + \frac{1}{\sigma} + \frac{1}{\rho}\left(\frac{dv}{ds} + \frac{u}{\sigma}\right) \end{aligned}\right\} \quad \text{......} \ (35).$$

The equations become

$$\left.\begin{aligned} &\frac{dN_1}{ds} - \frac{N_2}{\sigma} + \frac{T}{\rho} + X = 0, \\ &\frac{dN_2}{ds} \qquad + \frac{N_1}{\sigma} + Y = 0, \\ &\frac{dT}{ds} - \frac{N_1}{\rho} \qquad + Z = 0 \end{aligned}\right\} \quad \text{..................} (36),$$

and

$$\left.\begin{aligned}\frac{dG_1}{ds}-\frac{G_2}{\sigma}+\frac{H}{\rho}-N_2+L&=0,\\ \frac{dG_2}{ds}\qquad+\frac{G_1}{\sigma}+N_1+M&=0,\\ \frac{dH}{ds}-\frac{G_1}{\rho}\qquad\qquad+N&=0\end{aligned}\right\}\dots\dots\dots\dots\dots(37),$$

wherein

$$\left.\begin{aligned}G_1&=A\left[\frac{\beta}{\rho}-\frac{d}{ds}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)-\frac{1}{\sigma}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)\right],\\ G_2&=B\left[\frac{d}{ds}\left(\frac{d\mathrm{u}}{ds}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)-\frac{1}{\sigma}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)\right],\\ H&=C\left[\frac{d\beta}{ds}+\frac{1}{\rho}\left(\frac{d\mathrm{v}}{ds}+\frac{\mathrm{u}}{\sigma}\right)\right]\end{aligned}\right\}\dots(38),$$

and the condition of inextensibility is

$$\frac{d\mathrm{w}}{ds}-\frac{\mathrm{u}}{\rho}=0\ \dots\dots\dots\dots\dots\dots\dots\dots\ (39).$$

299. Equations of Small Motion.

The equations of small motion or free vibration will be found from the above by replacing the forces X, Y, Z and couples L, M, N by the reversed effective forces.

Instead of $-Xds$ we have to write the rate of change of momentum parallel to x of the element included between two normal sections distant ds. This is $\rho_0\omega ds\dfrac{\partial^2\mathrm{u}}{\partial t^2}$ where ω is the area of the normal section and ρ_0 is the density of the material. The quantities that replace Y and Z can be found in the same manner, and we have, corresponding to equations (36),

$$\left.\begin{aligned}\frac{\partial N_1}{\partial s}-\frac{N_2}{\sigma}+\frac{T}{\rho}&=\rho_0\omega\frac{\partial^2\mathrm{u}}{\partial t^2},\\ \frac{\partial N_2}{\partial s}\qquad+\frac{N_1}{\sigma}&=\rho_0\omega\frac{\partial^2\mathrm{v}}{\partial t^2},\\ \frac{\partial T}{\partial s}-\frac{N_1}{\rho}\qquad&=\rho_0\omega\frac{\partial^2\mathrm{w}}{\partial t^2}\end{aligned}\right\}\dots\dots\dots\dots\dots(40).$$

Instead of $-Lds$ we have to write the rate of change of the moment about the axis x of the momentum of the same element. This is $-\rho_0\omega dsk_2^2\dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial\mathrm{v}}{\partial s}+\dfrac{\mathrm{u}}{\sigma}\right)$. In like manner instead of $-Mds$

we have to write $\rho_0\omega ds k_1^2 \dfrac{\partial^2}{\partial t^2}\left(\dfrac{\partial \mathrm{u}}{\partial s} - \dfrac{\mathrm{v}}{\sigma} + \dfrac{\mathrm{w}}{\rho}\right)$, and instead of $-Nds$ we have to write $\rho_0\omega ds k^2 \dfrac{\partial^2\beta}{\partial t^2}$, where as in art. 262 k_2, k_1 and k are the radii of gyration of the normal section about the line-elements 1, 2 and 3. Hence we have the equations of small motion corresponding to (37),

$$\left.\begin{aligned}
&\frac{\partial G_1}{\partial s} - \frac{G_2}{\sigma} + \frac{H}{\rho} - N_2 = -\rho_0\omega k_2^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial \mathrm{v}}{\partial s} + \frac{\mathrm{u}}{\sigma}\right),\\
&\frac{\partial G_2}{\partial s} \qquad + \frac{G_1}{\sigma} + N_1 = \quad \rho_0\omega k_1^2 \frac{\partial^2}{\partial t^2}\left(\frac{\partial \mathrm{u}}{\partial s} - \frac{\mathrm{v}}{\sigma} + \frac{\mathrm{w}}{\rho}\right),\\
&\frac{\partial H}{\partial s} - \frac{G_1}{\rho} \qquad\qquad = \quad \rho_0\omega k^2 \frac{\partial^2\beta}{\partial t^2}
\end{aligned}\right\} \ldots (41).$$

In these equations

$$\left.\begin{aligned}
G_1 &= E\omega k_2^2\left[\frac{\beta}{\rho} - \frac{\partial}{\partial s}\left(\frac{\partial \mathrm{v}}{\partial s} + \frac{\mathrm{u}}{\sigma}\right) - \frac{1}{\sigma}\left(\frac{\partial \mathrm{u}}{\partial s} - \frac{\mathrm{v}}{\sigma} + \frac{\mathrm{w}}{\rho}\right)\right],\\
G_2 &= E\omega k_1^2\left[\frac{\partial}{\partial s}\left(\frac{\partial \mathrm{u}}{\partial s} - \frac{\mathrm{v}}{\sigma} + \frac{\mathrm{w}}{\rho}\right) - \frac{1}{\sigma}\left(\frac{\partial \mathrm{v}}{\partial s} + \frac{\mathrm{u}}{\sigma}\right)\right],\\
H &= \quad C\left[\frac{\partial\beta}{\partial s} + \frac{1}{\rho}\left(\frac{\partial \mathrm{v}}{\partial s} + \frac{\mathrm{u}}{\sigma}\right)\right]
\end{aligned}\right\} \ldots (42).$$

Also C is the torsional rigidity of the wire (cf. arts. 254, 264), and E is the Young's modulus of the material for pull in the direction of the elastic central-line.

The above equations with the condition of inextensibility (39) viz.

$$\frac{\partial \mathrm{w}}{\partial s} = \frac{\mathrm{u}}{\rho},$$

constitute the general equations of small vibration of the wire for those modes in which the extension of the elastic central-line may be disregarded.

There would be no difficulty in writing down for small motions the equations corresponding to those of art. 297 in which ϕ_0 is not zero.

300. Circular wire bent in its plane.

The simplest application of the equations of equilibrium will be found in the case of a naturally circular wire which if simply unbent would be prismatic, and which is bent in its own plane.

Suppose a is the radius of the circle which coincides with the elastic central-line of the wire in the unstrained state, and suppose that one principal axis of inertia of each normal section before strain coincides with a normal to the circle of radius a, and let θ be the angle between the radius to any point of the wire and a fixed radius; then we may replace ds by $ad\theta$.

Let u be the displacement along the normal inwards, and w the displacement along the tangent in the direction in which θ increases, then the condition of inextensibility is

$$\frac{d\mathrm{w}}{d\theta} = \mathrm{u} \qquad\qquad (43).$$

Since v, β and $1/\sigma$ all vanish, N_2, G_1 and H vanish, and the general equations (37) when there are no applied couples except at the ends become

$$\left.\begin{aligned} &\frac{dG_2}{d\theta} + N_1 a = 0, \\ &\frac{dN_1}{d\theta} + T + Xa = 0, \\ &\frac{dT}{d\theta} - N_1 + Za = 0 \end{aligned}\right\} \qquad\qquad (44),$$

where $Xad\theta$, $Zad\theta$ are the normal and tangential components of force on the element $ad\theta$.

By using (38) and (39) we have for the flexural couple

$$G_2 = \frac{B}{a^2}\left\{\frac{d^3\mathrm{w}}{d\theta^3} + \frac{d\mathrm{w}}{d\theta}\right\} \qquad\qquad (45).$$

From the first equation of (44) we have the shearing force N_1 given by

$$N_1 = -\frac{B}{a^3}\left(\frac{d^4\mathrm{w}}{d\theta^4} + \frac{d^2\mathrm{w}}{d\theta^2}\right) \qquad\qquad (46);$$

from the second equation of (44) we have the tension T given by

$$T = \frac{B}{a^3}\left(\frac{d^5\mathrm{w}}{d\theta^5} + \frac{d^3\mathrm{w}}{d\theta^3}\right) - Xa \qquad\qquad (47);$$

and from the third equation of (44) we have the differential equation for w

$$\frac{B}{a^3}\left[\frac{d^6\mathrm{w}}{d\theta^6} + 2\frac{d^4\mathrm{w}}{d\theta^4} + \frac{d^2\mathrm{w}}{d\theta^2}\right] + a\left(Z - \frac{dX}{d\theta}\right) = 0 \qquad\qquad (48).$$

This equation is identical for the case of equilibrium with that which has been given by Prof. Lamb (*Proc. Lond. Math. Soc.* XIX. 1888, p. 367). The following examples with the exception of 6° are given by him[1].

1°. When the bar is subjected to terminal couple $\overline{N}$ only, the central-line remains circular but its radius is reduced by the fraction $\overline{N}a/B$ of itself.

2°. When the wire is of length $2a\alpha$ and is subjected only to forces X along the chord joining its extremities, the displacements are given by

$$\text{w} = -a^3X\,\theta\,(\cos\alpha + \tfrac{1}{2}\cos\theta)/B,$$
$$\text{u} = -a^3X\,(\cos\alpha + \tfrac{1}{2}\cos\theta - \tfrac{1}{2}\,\theta\sin\theta)/B.$$

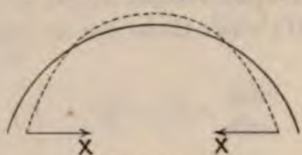


Fig. 45.

3°. When the bar is bent by equal and opposite forces Y applied at the extremities of rigid pieces attached to the ends the displacement w is given by

$$\text{w} = \tfrac{1}{2}a^3Y\theta\sin\theta/B.$$

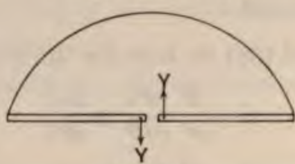


Fig. 46.

4°. A circular hoop is subjected to normal forces X at the extremities of a diameter.

When $\pi > \theta > 0$

$$\text{w} = \frac{Xa^3}{2B}\left(1 - \frac{2\theta}{\pi} - \cos\theta - \tfrac{1}{2}\,\theta\sin\theta\right),$$

and when $-\pi < \theta < 0$

$$\text{w} = \frac{Xa^3}{2B}\left(-1 - \frac{2\theta}{\pi} + \cos\theta + \tfrac{1}{2}\,\theta\sin\theta\right).$$

[1] The results only are here stated, and the student is recommended to supply the necessary analysis.

Fig. 47.

The corresponding values of u are easily written down.

It may be readily deduced that, as noticed by Saint-Venant[1], the diameter $\theta=0$ is shortened by $\dfrac{Xa^3(\pi^2-8)}{4\pi B}$, while the perpendicular diameter is increased by $\dfrac{Xa^3(4-\pi)}{2\pi B}$.

5°. When a circular hoop of weight W is suspended from a point in its circumference

$$\text{w}=-\frac{Wa^3}{8\pi B}\{(\theta-\pi)^2\sin\theta+4(\theta-\pi)\cos\theta-4(\theta-\pi)-\pi^2\sin\theta\},$$

θ being measured from the highest point. By comparison with the preceding it appears that the increase in the vertical diameter and the shortening of the horizontal diameter are each half what they would be if the weight W were concentrated at the lowest point.

6°. A circular hoop of mass m per unit of length rotates round one diameter which is taken as axis of y (see fig. 48) with angular velocity ω, one extremity of that diameter being fixed. Its central-line describes a surface of revolution about the axis y whose meridian curve is given by the equations[2]

$$x=a\sin\theta+\frac{m\omega^2a^5\sin^3\theta}{12B},$$

$$y=a(1-\cos\theta)-\frac{m\omega^2a^5(1-\cos^3\theta)}{12B},$$

θ being measured from the centre and the vertical diameter.

[1] *Mémoires sur la Résistance des Solides....* Paris, 1844.

[2] G. A. V. Peschka. 'Ueber die Formveränderungen prismatischer Stäbe durch Biegung'. Schlömilch's *Zeitschrift*, XIII. 1868.

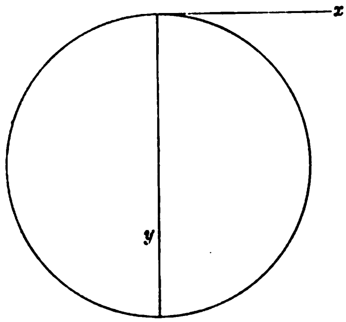


Fig. 48.

The diameter (marked y in the figure) about which the wire rotates is shortened, and the perpendicular diameter lengthened by the same amount $\frac{1}{8}m\omega^2a^5/B$.

301. Circular wire bent perpendicularly to its plane.

As a simple example involving displacements not all in one plane we may consider the case of a naturally circular wire bent by forces applied perpendicularly to the plane of the circle[1].

Suppose the wire supported at one end and its plane horizontal, and suppose a weight W attached to the other end.

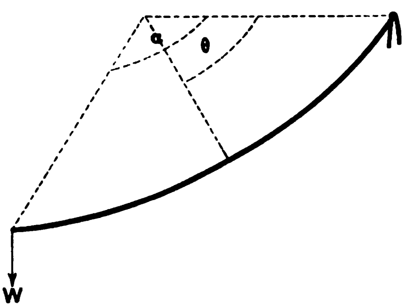


Fig. 49.

Let a be the radius of the circle, and θ the angle between the radius vector drawn to any point of the elastic central-line before

[1] Resal, Liouville's *Journal*, III. 1877, and Saint-Venant, *Comptes Rendus*, XVII. 1843, pp. 1023—1031.

strain and that drawn to the point of support, and suppose the weight W attached at the point $\theta = \alpha$.

The equations of equilibrium become

$$\left.\begin{aligned} \frac{dN_1}{d\theta} + T &= 0, \\ \frac{dN_2}{d\theta} \qquad &= 0, \\ \frac{dT}{d\theta} - N_1 &= 0 \end{aligned}\right\} \text{.........................(49)},$$

and

$$\left.\begin{aligned} \frac{dG_1}{d\theta} + H - N_2 a &= 0, \\ \frac{dG_2}{d\theta} \qquad + N_1 a &= 0, \\ \frac{dH}{d\theta} - G_1 \qquad &= 0 \end{aligned}\right\} \text{.................. (50)}.$$

The conditions at the loaded end are $N_1 = 0$, $N_2 = W$, $T = 0$, and $G_1 = 0$, $G_2 = 0$, $H = 0$.

From equations (49) we find

$$N_2 = \text{const.} = W,$$

$$N_1 = 0, \text{ and } T = 0;$$

so that the first and third of equations (50) become

$$\left.\begin{aligned} \frac{dG_1}{d\theta} + H - Wa &= 0, \\ \frac{dH}{d\theta} - G_1 \qquad &= 0 \end{aligned}\right\} \text{.................. (51)}.$$

Hence $$\frac{d^2 G_1}{d\theta^2} + G_1 = 0;$$

so that we have

$$\left.\begin{aligned} G_1 &= Wa \sin(\alpha - \theta), \\ H &= Wa\{1 - \cos(\alpha - \theta)\} \end{aligned}\right\} \text{................ (52)},$$

where H is found from the first of equations (51), and the constants have been chosen so as to satisfy the conditions at the loaded end.

Now, by (38), these equations give us

$$\left.\begin{aligned}\frac{d^2\mathrm{v}}{d\theta^2} - \quad a\beta &= -\frac{Wa^3}{A}\sin(\alpha-\theta),\\ \frac{d\mathrm{v}}{d\theta} + a\frac{d\beta}{d\theta} &= \quad\frac{Wa^3}{C}\{1-\cos(\alpha-\theta)\}\end{aligned}\right\}\ \ldots\ldots\ldots\ (53),$$

where β is the angle defining the twist, and v the vertical displacement downwards. We deduce the equation for v

$$\frac{d^3\mathrm{v}}{d\theta^3} + \frac{d\mathrm{v}}{d\theta} = \frac{Wa^3}{C} - Wa^3\left(\frac{1}{C}-\frac{1}{A}\right)\cos(\alpha-\theta)\ \ldots\ldots\ (54),$$

and when v is found from this β is given by the first of (53).

The horizontal displacements u and w are given by the equations

$$\frac{dG_2}{d\theta} = 0,$$

$$G_2 = \frac{B}{a^2}\left(\frac{d^2\mathrm{u}}{d\theta^2} + \mathrm{u}\right),$$

$$\frac{d\mathrm{w}}{d\theta} = \mathrm{u},$$

and the terminal conditions at $\theta = \alpha$ require that $G_2 = 0$, and then from the terminal conditions at $\theta = 0$ we find that u and w both vanish.

It appears from (23) of art. 294 that the left-hand member of equation (54) is a^2/σ', where $1/\sigma'$ is the measure of tortuosity of the curve into which the elastic central-line is deformed. Equation (54) is identical with one obtained by M. Resal.

The solution of this equation involves three arbitrary constants which can be determined from the conditions that hold at the fixed point $\theta = 0$. If we suppose that at this point the tangent and normal are fixed in direction we find, as the terminal conditions, $\mathrm{v} = 0$, $d\mathrm{v}/d\theta = 0$, $\beta = 0$ when $\theta = 0$. The last is derived from the expression for the direction-cosines of the line initially coinciding with the principal normal given in (27) of art. 294, or directly by considering the meaning of β.

It will be found that the vertical displacement v is given by the equation

$$\mathrm{v} = \frac{Wa^3}{C}\{(\theta - \sin\theta) - \sin\alpha(1-\cos\theta)\}$$
$$+ \tfrac{1}{2} Wa^3\left(\frac{1}{C} - \frac{1}{A}\right)\{\theta\cos(\alpha-\theta) - \sin\theta\cos\alpha\} \ldots (55)^1.$$

If the wire be of isotropic material and circular section, the radius of the section being c, then $A = \frac{1}{4}\pi c^4 E$ and $C = \frac{1}{2}\pi c^4 \mu$, where E is the Young's modulus and μ the rigidity of the material.

302. Vibrations of Circular Wire.

As an example of the application of the theory to vibrations let us consider the small free vibrations of a naturally circular wire of circular section. Let c be the radius of the normal section, and a the radius of the circle formed by the elastic central-line in the unstrained state. We may replace ρ in the equations of art. 299 by a, and ds by $ad\theta$. Also we have to write

$$k_1^2 = k_2^2 = \tfrac{1}{4}c^2, \; k^2 = \tfrac{1}{2}c^2,$$

and $\omega = \pi c^2$. Thus our equations become

$$\left.\begin{aligned} \frac{\partial N_1}{\partial\theta} + T &= \pi\rho_0 c^2 a \frac{\partial^2 \mathrm{u}}{\partial t^2}, \\ \frac{\partial N_2}{\partial\theta} \qquad &= \pi\rho_0 c^2 a \frac{\partial^2 \mathrm{v}}{\partial t^2}, \\ \frac{\partial T}{\partial\theta} - N_1 &= \pi\rho_0 c^2 a \frac{\partial^2 \mathrm{w}}{\partial t^2}, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (56),$$

and
$$\left.\begin{aligned} \frac{\partial G_1}{\partial\theta} + H - N_2 a &= -\tfrac{1}{4}\pi\rho_0 c^4 \frac{\partial^3 \mathrm{v}}{\partial t^2 \partial\theta}, \\ \frac{\partial G_2}{\partial\theta} \qquad + N_1 a &= \tfrac{1}{4}\pi\rho_0 c^4\left(\frac{\partial^3 \mathrm{u}}{\partial t^2\partial\theta} - \frac{\partial^2 \mathrm{w}}{\partial t^2}\right), \\ \frac{\partial H}{\partial\theta} - G_1 \qquad &= \tfrac{1}{2}\pi\rho_0 c^4 a \frac{\partial^2\beta}{\partial t^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots (57),$$

wherein
$$\left.\begin{aligned} G_1 &= \tfrac{1}{4}E\pi\frac{c^4}{a^2}\left(a\beta - \frac{\partial^2 \mathrm{v}}{\partial\theta^2}\right), \\ G_2 &= \tfrac{1}{4}E\pi\frac{c^4}{a^2}\left(\frac{\partial^2 \mathrm{u}}{\partial\theta^2} + \frac{\partial \mathrm{w}}{\partial\theta}\right), \\ H &= \tfrac{1}{2}\mu\pi\frac{c^4}{a^2}\left(\frac{\partial \mathrm{v}}{\partial\theta} + a\frac{\partial\beta}{\partial\theta}\right) \end{aligned}\right\} \ldots\ldots\ldots\ldots (58),$$

[1] This does not agree with the result given by M. Resal, but I do not understand his analysis. For $\alpha = \frac{1}{2}\pi$ it agrees with the result given by Saint-Venant.

with the condition of inextensibility

$$\partial \mathrm{w}/\partial \theta = \mathrm{u} \quad\text{..............................} \quad (59).$$

These equations can be separated into two sets[1], viz. the condition (59) with the first and third of (56) and the second of (57) and (58) form a set of equations connecting u, w, G_2, N_1, and T, while the second of (56) with the first and third of (57) and (58) form a set of equations connecting v, β, G_1, H, and N_2.

303. Flexural vibrations in the plane of the circle.

The equations connecting u, w, G_2, N_1, T can be reduced to the forms

$$\frac{\partial^2 N_1}{\partial \theta^2} + N_1 = \pi \rho_0 c^2 a \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \theta^2} - \mathrm{w}\right),$$

$$\tfrac{1}{4} E\pi \frac{c^4}{a^2}\left(\frac{\partial^4 \mathrm{w}}{\partial \theta^4} + \frac{\partial^2 \mathrm{w}}{\partial \theta^2}\right) + N_1 a = \tfrac{1}{4}\pi \rho_0 c^4 \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \theta^2} - \mathrm{w}\right).$$

If we neglect "rotatory inertia[2]" we must reject the right-hand member of the second of these equations; and then it is easy to eliminate N_1, and obtain the equation for w.

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \theta^2} - \mathrm{w}\right) + \tfrac{1}{4}\frac{Ec^2}{\rho_0 a^4}\left(\frac{\partial^6 \mathrm{w}}{\partial \theta^6} + 2\frac{\partial^4 \mathrm{w}}{\partial \theta^4} + \frac{\partial^2 \mathrm{w}}{\partial \theta^2}\right) = 0 \quad \text{...}(60).$$

Supposing $\mathrm{w} \propto e^{ipt}$, so that $2\pi/p$ is the frequency, and writing

$$m^2 = 4p^2\rho_0 a^4/Ec^2 \quad\text{........................}(61),$$

we have

$$\frac{\partial^6 \mathrm{w}}{\partial \theta^6} + 2\frac{\partial^4 \mathrm{w}}{\partial \theta^4} + (1 - m^2)\frac{\partial^2 \mathrm{w}}{\partial \theta^2} + m^2 \mathrm{w} = 0 \quad\text{............}(62).$$

The solution of this equation may be written in the form

$$\mathrm{w} = A_1 \cos n_1\theta + A_2 \cos n_2\theta + A_3 \cos n_3\theta + B_1 \sin n_1\theta + \dots,$$

where n_1^2, n_2^2, n_3^2 are the roots of the equation

$$n^2(n^2 - 1)^2 - (n^2 + 1)m^2 = 0 \quad\text{..................}(63).$$

[1] The same thing holds good whatever the initial form of the wire may be, provided only its elastic central-line is a plane curve in a principal plane of the wire.

[2] There is no difficulty in the mathematical work when 'rotatory inertia' is retained, but the results are somewhat more complicated and the physical interest is diminished. The student may work out the correction for 'rotatory inertia'.

When the wire forms a complete circular ring[1] n is an integer, and then the frequency-equation is

$$p^2 = \tfrac{1}{4}\frac{Ec^2}{\rho_0 a^4}\frac{n^2(n^2-1)^2}{1+n^2} \quad \text{.................. (64).}$$

The number n is the number of wave-lengths in the circumference of the ring. The frequency is of the same order of magnitude as in a straight bar of the same section whose length is equal to half the circumference. For the modes of low pitch the sequence of the component tones in the two cases is quite different, but when n is great the frequencies tend to become identical.

When the wire does not form a complete circle it is convenient to take the origin of θ at its middle point. Considering the case where the wire subtends an angle 2α at the centre of the circle, and has free ends, we shall have at either end $G_2 = 0$, $N_1 = 0$, $T = 0$, so that at $\theta = \pm\alpha$ we have

$$\left.\begin{aligned} &\frac{\partial^3 \mathrm{w}}{\partial\theta^3} + \frac{\partial \mathrm{w}}{\partial\theta} = 0, \\ &\frac{\partial^4 \mathrm{w}}{\partial\theta^4} + \frac{\partial^2 \mathrm{w}}{\partial\theta^2} = 0, \\ &\frac{\partial^5 \mathrm{w}}{\partial\theta^5} + \frac{\partial^3 \mathrm{w}}{\partial\theta^3} - m^2\frac{\partial \mathrm{w}}{\partial\theta} = 0 \end{aligned}\right\} \quad \text{.................. (65),}$$

and in virtue of the first of these and the differential equation (62) the last may be written

$$\int \mathrm{w}\, d\theta = 0.$$

The fundamental modes[2] fall into two classes according as w is an odd or an even function of θ. Taking first the case where w is an odd function the coefficients A vanish, and the coefficients B are connected by three linear relations which are easily written down. The elimination of the B's leads to an equation which may be shewn to be

$$n_1^3(1-n_1^4)(n_2^2-n_3^2)\tan n_1\alpha + n_2^3(1-n_2^4)(n_3^2-n_1^2)\tan n_2\alpha$$
$$+ n_3^3(1-n_3^4)(n_1^2-n_2^2)\tan n_3\alpha = 0 \text{......(66).}$$

This is really an equation to find m.

[1] Cf. Hoppe in Crelle-Borchardt, LXXIII. 1871, and Lord Rayleigh's *Theory of Sound*, vol. I. p. 324.

[2] The work of verifying this and the following statements is left to the reader.

In like manner, when w is an even function the coefficients B vanish, and the A's are connected by three linear relations. If these be written down, and the A's eliminated, the n's will be connected by an equation which may be shewn to be

$$n_1^3(1-n_1^4)(n_2^2-n_3^2)\cot n_1\alpha+n_2^3(1-n_2^4)(n_3^2-n_1^2)\cot n_2\alpha$$
$$+n_3^3(1-n_3^4)(n_1^2-n_2^2)\cot n_3\alpha=0 \ldots\ldots(67),$$

and this is really an equation to find m.

For a discussion of the results the reader is referred to Prof. Lamb's paper 'On the Flexure and Vibrations of a Curved Bar' in *Proc. Lond. Math. Soc.* XIX. 1888. When the curvature is very slight the motion is nearly the same as for a straight bar of the same length, but the pitch is slightly lowered.

304. Flexural vibrations perpendicular to the plane of the circle.

The equations of art. 302 connecting v, β, G_1, H, and N_2 can be reduced to the forms

$$\left.\begin{array}{r}\frac{1}{4}E\pi\frac{c^4}{a^2}\left(a\frac{\partial^2\beta}{\partial\theta^2}-\frac{\partial^4 \mathrm{v}}{\partial\theta^4}\right)+\frac{1}{2}\mu\pi\frac{c^4}{a^2}\left(\frac{\partial^2 \mathrm{v}}{\partial\theta^2}+a\frac{\partial^2\beta}{\partial\theta^2}\right)\qquad\qquad\\ -\pi\rho_0c^2a^2\frac{\partial^2 \mathrm{v}}{\partial t^2}+\frac{1}{4}\pi\rho_0c^4\frac{\partial^4 \mathrm{v}}{\partial t^2\partial\theta^2}=0,\\ \frac{1}{4}E\pi\frac{c^4}{a^2}\left(a\beta-\frac{\partial^2 \mathrm{v}}{\partial\theta^2}\right)-\frac{1}{2}\mu\pi\frac{c^4}{a^2}\left(\frac{\partial^2 \mathrm{v}}{\partial\theta^2}+a\frac{\partial^2\beta}{\partial\theta^2}\right)+\frac{1}{2}\pi\rho_0c^4a\frac{\partial^2\beta}{\partial t^2}=0\end{array}\right\}\ldots(68).$$

To satisfy these equations we assume

$$\mathrm{v}=Ae^{\iota(n\theta+pt)},\quad a\beta=Be^{\iota(n\theta+pt)} \ldots\ldots\ldots\ldots (69),$$

so that $p/2\pi$ is the frequency. Then A and B are connected by the relations

$$\left.\begin{array}{r}\left[\rho_0p^2(4a^2+n^2c^2)-(En^2+2\mu)\frac{c^2}{a^2}n^2\right]A-(E+2\mu)\frac{c^2}{a^2}n^2B=0,\\ -(E+2\mu)\frac{n^2}{a^2}A+\left[2\rho_0p^2-(E+2\mu n^2)\frac{1}{a^2}\right]B=0\end{array}\right\}(70).$$

On eliminating A and B we obtain the frequency-equation[1]

$$8\rho_0^2p^4a^4\left(1+\frac{1}{4}n^2\frac{c^2}{a^2}\right)$$
$$-\rho_0p^2a^2\left[4(E+2\mu n^2)+\frac{n^2c^2}{a^2}\{E(1+2n^2)+2\mu(2+n^2)\}\right]$$
$$+2\frac{n^2c^2}{a^2}(n^2-1)^2E\mu=0 \ldots\ldots\ldots\ldots (71).$$

[1] This equation was first given by Basset. *Proc. Lond. Math. Soc.* XXIII. 1892.

This equation gives two distinct values of p^2, and the vibrations involving v and β are of two distinct types. We may approximate to the roots by regarding c^2/a^2 as a small quantity, and then we easily find that there is a vibration of short period with a frequency given by

$$p_1^2 = \tfrac{1}{2}\frac{E+2\mu n^2}{\rho_0 a^2} \quad\text{.....................} (72),$$

and a vibration of long period with a frequency given by

$$p_2^2 = \tfrac{1}{2}\frac{E\mu c^2 n^2 (n^2-1)^2}{\rho_0 a^4 (E+2\mu n^2)} \quad\text{...............}(73)^1.$$

If we substitute p_1^2 for p^2 we see from the second of equations (70) that A vanishes, and therefore the vibrations of short period involve no displacement of the elastic central-line. To a higher order of approximation the displacement v perpendicular to the plane of the circle is of the order c^2/a^2 in comparison with the displacement $a\beta$ due to the torsion. The vibrations of short period are therefore almost purely torsional. We shall return to the discussion of these vibrations presently.

The vibrations of long period involve flexure perpendicular to the plane of the circle. We shall suppose the wire forms a complete circular ring[2]. In that case the quantity n must be an integer, and it is the number of wave-lengths in the circumference when any normal vibration is being executed. Remembering the relation $E = 2\mu(1+\sigma)$ connecting the Young's modulus, the rigidity, and the Poisson's ratio of isotropic material, we may rewrite the frequency equation (73) in the form

$$p_2^2 = \tfrac{1}{4}\frac{Ec^2}{\rho_0 a^4}\frac{n^2(n^2-1)^2}{1+\sigma+n^2} \quad\text{..................} (74).$$

Comparing this result with that in equation (64) of the last article, and remembering that σ for most hard solids does not differ much from $\frac{1}{4}$, while n is an integer not less than 2, we see that the frequencies of the various modes involving flexure perpendicular to the plane of the ring are almost identical with the frequencies of the corresponding modes involving flexure in the plane of the ring. The difference, even in the case of the gravest

[1] A result equivalent to this was obtained by Michell. *Messenger of Mathematics*, XIX. 1889.

[2] The consideration of the case of free ends may serve as an exercise for the student.

mode, is such as would be neglected in tuning by "equal temperament", and for the higher modes it may be completely disregarded.

It is most important to observe that the modes of vibration which involve flexure perpendicular to the plane of the ring involve also a twist of amount comparable with the flexure. The twist is in fact $\frac{1}{a^2}\left(a\frac{\partial\beta}{\partial\theta}+\frac{\partial \mathrm{v}}{\partial\theta}\right)$, and the change of curvature is $\frac{1}{a^2}\left(a\beta-\frac{\partial^2\mathrm{v}}{\partial\theta^2}\right)$. To the first order of small quantities, this change of curvature is not a change in the magnitude of the curvature in the osculating plane but is the expression of a periodic change of adjustment of the osculating plane with reference to lines of the material and with reference to lines fixed in space. The feature that distinguishes the modes in question from other modes of which the ring is capable is the displacement of a point on the elastic central-line perpendicular to the plane of the circle, and we have thought that this characteristic is better expressed by a reference to the flexure than by a reference to the torsion.

305. Torsional and extensional vibrations.

The torsional vibrations have already been partly considered. If we suppose that v is of the order c^2/a^2 compared with $a\beta$, and reject terms of this order, we shall have an approximate solution of equations (68) in the form

$$\mathrm{v}=0, \quad a\beta=\Sigma B_n\cos(n\theta+\alpha)\,e^{\iota p_1 t} \;\ldots\ldots\ldots\ldots (75),$$

so that the elastic central-line remains fixed while the sections rotate about it. The frequency $p_1/2\pi$ is given by the equation (72), which we may write

$$p_1^2=\frac{\mu}{\rho_0 a^2}(1+\sigma+n^2) \;\ldots\ldots\ldots\ldots\ldots\ldots (76).$$

The modes $n=0$, $n=1$ are the most interesting. When $n=0$ or β is independent of θ, all the normal sections are at any instant rotated through the same angle about the elastic central-line. There is in the technical sense no "twist", but the vibrations depend on the extensions and contractions accompanying a continual change of adjustment of the plane of flexure with reference

to the material. The frequency of such vibrations is given by $p^2 = E/2\rho_0 a^2$.

When $n=1$ the wave-length is equal to the circumference of the circle, and the frequency is given by $p^2 = \mu(2+\sigma)/\rho_0 a^2$.

All the remaining modes of torsional vibration are of very high pitch as compared with the corresponding modes of flexural vibration, and those corresponding to $n=0$ and $n=1$ are also of high pitch as compared with the graver modes of flexural vibration. The frequency of any mode is of the same order of magnitude as the frequency of torsional vibrations of a straight bar of the same material and of length equal to half the circumference, and for the higher modes the sequences of tones in the cases are ultimately identical.

The *extensional vibrations* of a wire of any form can be investigated from the general equations of art. 299 by supposing that the couples and shearing forces vanish while the tension is $E\omega\left(\frac{\partial \mathrm{w}}{\partial s} - \frac{\mathrm{u}}{\rho}\right)$. Taking the case of a circular wire the equations (56) of art. 302 give us

$$\left.\begin{aligned} \frac{\partial^2 \mathrm{u}}{\partial t^2} &= \frac{E}{\rho_0 a^2}\left(\frac{\partial \mathrm{w}}{\partial \theta} - \mathrm{u}\right), \\ \frac{\partial^2 \mathrm{w}}{\partial t^2} &= \frac{E}{\rho_0 a^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \theta^2} - \frac{\partial \mathrm{u}}{\partial \theta}\right) \end{aligned}\right\} \quad \text{...............} (77).$$

Taking u and w proportional to $e^{\iota pt}$, so that $2\pi/p$ is the frequency, and writing

$$n^2 + 1 = \frac{p^2 \rho_0 a^2}{E},$$

we find without difficulty that u and w are of the forms

$$\left.\begin{aligned} \mathrm{u} &= n^{-1}(A\sin n\theta + B\cos n\theta)\, e^{\iota pt}, \\ \mathrm{w} &= \phantom{n^{-1}}(A\cos n\theta - B\sin n\theta)\, e^{\iota pt}, \end{aligned}\right\} \quad \text{............} (78).$$

In the case of a complete circular ring[1] n must be an integer, and the frequency when there are n wave-lengths to the circumference is given by the equation

$$p^2 = \frac{E}{\rho_0 a^2}(1+n^2) \quad \text{......................} (79).$$

[1] The case of a circular wire with free ends is left to the reader.

The case $n = 0$ corresponds to purely *radial vibrations*. For these u is independent of θ while w vanishes, and the frequency is given by

$$p^2 = \frac{E}{\rho_0 a^2} \qquad (80).$$

The frequencies of the extensional modes are of the same order of magnitude as in a bar of length equal to half the circumference, and the sequences of tones tend to become identical in the two cases as the number of wave-lengths increases.

It will be observed that the vibrations we have described as "extensional" involve a change of curvature, and are therefore in some sense "flexural". The curvature of the elastic central-line is in fact increased by

$$\frac{1}{a^2}\left(\frac{\partial^2 \mathrm{u}}{\partial \theta^2} + \frac{\partial \mathrm{w}}{\partial \theta}\right),$$

and, when the wire is free at $\theta = 0$ and $\theta = \alpha$, this is

$$-\frac{2}{a^2} \Sigma \frac{s\pi}{\alpha} A_s \sin \frac{s\pi\theta}{\alpha} e^{\iota pt},$$

where s is an integer, which is sufficiently general to express any given initial condition as to curvature possible with free ends.

If a wire be struck or otherwise thrown into vibration it becomes a question whether it will take up a "flexural" or an "extensional" mode, and we can be guided to a correct answer by general principles. For, in the first place, if two modes of deformation be possible involving the same initial changes of curvature, but of which one involves no extension of the elastic central-line, the potential energy of the deformed wire in the flexural mode will have ultimately to that in the extensional mode a ratio which is of the second order in the small quantity (diameter of section) $\div$ (diameter of circle), and the system will tend to take up that mode of vibration in which the potential energy is less. Again, the periods of the extensional modes are proportional to the diameter of the circle formed by the unstrained elastic central-line, while the periods of the flexural modes are proportional to the square of the diameter of this circle and inversely proportional to the diameter of the section. The flexural modes are therefore much graver than the extensional, and it is a general principle that vibrations of low pitch are more easily excited than vibrations

of high pitch. For example, it is difficult, except in carefully performed experiments, to make a straight rod vibrate "longitudinally" without setting up a "lateral vibration" which tends to become predominant.

The conclusion with regard to extensional vibrations of a circular wire is that somewhat extreme conditions would be necessary in order to set them up. For example, purely radial vibrations of a complete circular ring are possible, but any deviation from uniformity in the initial displacement will tend to introduce flexural vibrations of much lower pitch than the radial vibrations.

CHAPTER XIX.

ELEMENTARY THEORY OF THIN PLATES.

306. In the present chapter we propose to consider a theory of the infinitesimal flexure of a thin elastic plate. The general theory of the deformation of plates is difficult, but the particular modes of deformation which we shall here discuss admit of comparatively simple treatment, and they at the same time include most of those which are practically important.

Consider a thin sheet of homogeneous isotropic elastic matter in its natural state, bounded by parallel planes and by a cylindrical surface cutting them at right angles. Such a body is a thin plate; the plane bounding surfaces are called the *faces*; the plane midway between them, the *middle-surface*; the cylindrical bounding surface, the *edge*; and the line in which the edge cuts the middle-surface, the *edge-line*.

When the plate undergoes a small deformation the particles originally on the middle-surface come to lie on a curved surface which is everywhere very nearly plane. This surface will be called the *strained middle-surface*. The problem with which we shall be occupied is the determination of the form of this surface when the plate vibrates or is deformed by given forces.

Now in a very thin plate it is clear that it can make no sensible difference to the form of the strained middle-surface whether the forces that produce bending are applied as tension or pressure to its faces or as bodily force acting on a small volume, and we shall found the theory on the principle that the external forces applied to an element of the plate bounded by a slender prism normal to

its faces are sufficiently represented by their force- and couple-resultants[1]. The same applies to the stress across a section normal to the middle-surface, and to the applied forces on an element of the boundary.

307. Stress-system.

Suppose now that the plate is slightly deformed. Let us take a system of axes of x, y, z, of which z is the normal to the middle-surface before strain, and suppose that the plane faces are given by $z = \pm h$, so that $2h$ is the thickness of the plate, and consider the stress across a normal section.

Let P, Q, R, S, T, U be the six components of stress at any point. We have to reduce the stress across normal sections perpendicular to the axes of x and y to force- and couple-resultants.

Across an element of a normal section perpendicular to x of length dy and breadth $2h$ the stress-resultants are such quantities as

$$dy \int_{-h}^{h} P dz \text{ parallel to } x,$$

and the stress-couples are such quantities as

$$dy \int_{-h}^{h} -zU dz \text{ about } x;$$

all these quantities contain dy as a factor, and the other factors represent the stress-resultants and stress-couples per unit length of the trace of the normal section on the middle-surface.

If we write

$$\int_{-h}^{h} P dz = P_1, \quad \int_{-h}^{h} Q dz = P_2, \quad \int_{-h}^{h} S dz = T_2,$$
$$\int_{-h}^{h} T dz = T_1, \quad \int_{-h}^{h} U dz = U_1 \ldots\ldots(1),$$

and

$$\int_{-h}^{h} Pz dz = G_1, \quad \int_{-h}^{h} - Qz dz = G_2, \quad \int_{-h}^{h} - Uz dz = H \ldots(2),$$

then P_1, U_1, T_1 are the stress-resultants parallel to the axes on the face $x = \text{const.}$, U_1, P_2, T_2 are the stress-resultants on the face

[1] Cf. art. 249. See also Lamb 'On the Flexure of an Elastic Plate'. *Proc. Lond. Math. Soc.* XXI. 1890.

$y = \text{const.}$, H and G_1 are the stress-couples about the axes of x and y on the face $x = \text{const.}$, and G_2 and $-H$ are the stress-couples about the axes of x and y on the face $y = \text{const.}$, and each of these is estimated per unit length of the trace on the middle-surface of the plane across which it acts.

308. Equations of Equilibrium.

Consider the equilibrium of an element of the plate bounded by its faces and by four planes x, $x + dx$, y, $y + dy$, which is subject to the action of forces normal to the middle-surface and couples about axes parallel to the middle-surface.

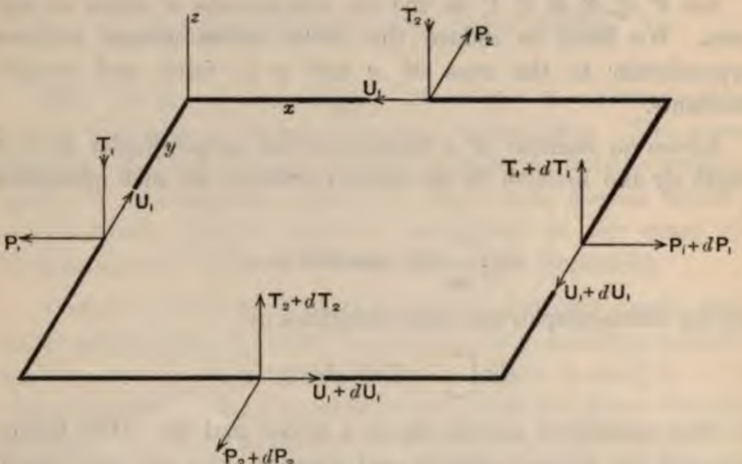


Fig. 50.

Let the external forces applied to the element be reduced to a force at $(x, y, 0)$ of amount $2Zhdxdy$ parallel to the axis z, and couples $2Lhdxdy$, $2Mhdxdy$ about the axes x and y.

We can put down the components (parallel to the axes of x, y, z) of the stress-resultants on the four faces.

On the face x we have

$$-P_1 dy, \quad -U_1 dy, \quad -T_1 dy \qquad\qquad (3),$$

parallel to the axes, and on the face $x + dx$ we have

$$P_1 dy + \frac{\partial}{\partial x}(P_1 dy)\, dx, \qquad U_1 dy + \frac{\partial}{\partial x}(U_1 dy)\, dx,$$

$$T_1 dy + \frac{\partial}{\partial x}(T_1 dy)\, dx \qquad\qquad (4).$$

On the face y we have

$$-U_1dx, \quad -P_2dx, \quad -T_2dx \qquad \text{(5)},$$

and on the face $y+dy$ we have

$$U_1dx + \frac{\partial}{\partial y}(U_1dx)\,dy, \qquad P_2dx + \frac{\partial}{\partial y}(P_2dx)\,dy,$$

$$T_2dx + \frac{\partial}{\partial y}(T_2dx)\,dy \qquad \text{(6)}.$$

Adding together all the forces parallel to x, y, or z, and equating the sum in each case to zero, we obtain the three equations of resolution

$$\left.\begin{aligned} \frac{\partial P_1}{\partial x} + \frac{\partial U_1}{\partial y} &= 0, \\ \frac{\partial U_1}{\partial x} + \frac{\partial P_2}{\partial y} &= 0, \\ \frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y} + 2hZ &= 0 \end{aligned}\right\} \qquad \text{(7)}.$$

In like manner we can put down the couples about axes parallel to the axes of x and y that act on the four faces.

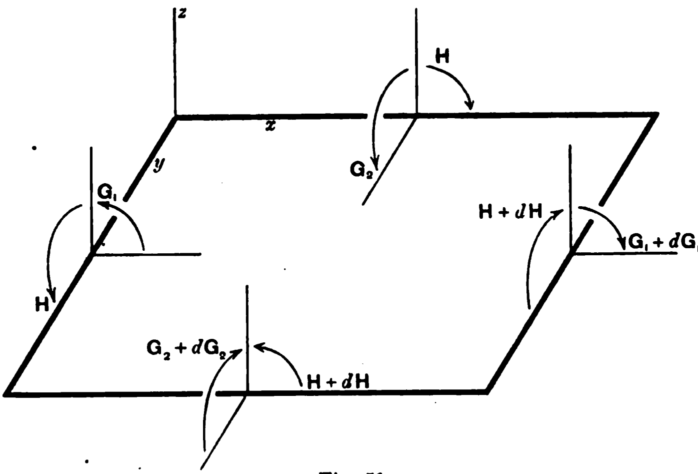


Fig. 51.

On the face x we have

$$-Hdy, \quad -G_1dy,$$

and on the face $x+dx$ we have

$$Hdy + \frac{\partial}{\partial x}(Hdy)\,dx, \qquad G_1dy + \frac{\partial}{\partial x}(G_1dy)\,dx.$$

On the face y we have

$$-G_2 dx, \quad H dx,$$

and on the face $y + dy$ we have

$$G_2 dx + \frac{\partial}{\partial y}(G_2 dx)\, dy, \qquad -H dx - \frac{\partial}{\partial y}(H dx)\, dy.$$

Also the systems (4) and (6) give us moments

$$T_2 dx dy, \quad -T_1 dx dy,$$

about axes parallel to the axes of x and y drawn through the point $(x, y, 0)$.

Hence the equations of moments about the axes of x and y are

$$\left.\begin{aligned} \frac{\partial H}{\partial x} + \frac{\partial G_2}{\partial y} + T_2 + 2hL = 0, \\ \frac{\partial G_1}{\partial x} - \frac{\partial H}{\partial y} - T_1 + 2hM = 0 \end{aligned}\right\} \text{..................(8).}$$

The equations (7) and (8) are the general equations of equilibrium of the plate when slightly deformed by normal forces and couples.

The equations for P_1, P_2, U_1 and those for T_1, T_2, G_1, G_2, H are quite independent, and the latter are the equations on which the flexure of the plate depends. If we eliminate T_1 and T_2 from equations (8) and the third of equations (7), we obtain the equation

$$-\frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_2}{\partial y^2} + 2\frac{\partial^2 H}{\partial x \partial y} = 2h\left[Z + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right] \text{.........(9),}$$

which is the general equation for the equilibrium of a bent plate.

309. Internal Strain[1].

Consider a state of strain in the plate characterised by the following properties:

(*a*) the middle-surface is unextended,

(*b*) line-elements of the plate initially normal to the middle-surface remain straight and normal to the middle-surface after strain,

(*c*) there is no normal traction across planes parallel to the middle-surface.

[1] Cf. Lamb, *Proc. Lond. Math. Soc.* xxi. 1890, and Saint-Venant's 'Annotated Clebsch' *note du* § 73.

Let w be the deflexion at any point of the plate, *i.e.* w is to denote the displacement parallel to the axis z of any particle of the plate initially on the middle-surface. The condition (a) will be satisfied if the displacement of the particle parallel to the middle-surface be zero, and if w be so small that we may neglect its square.

If x, y, z be the coordinates of a particle of the plate before strain, and x', y', z' the coordinates of the same particle after strain, we shall be able to satisfy the condition (b) by taking

$$x' = x - z\frac{\partial \mathrm{w}}{\partial x}, \qquad y' = y - z\frac{\partial \mathrm{w}}{\partial y},$$

and then $x'-x$, $y'-y$ are the component displacements of the particle parallel to the axes of x and y. Of the six strains e, f, g, a, b, c, three, viz. e, f, c are given by

$$e = -z\frac{\partial^2 \mathrm{w}}{\partial x^2}, \qquad f = -z\frac{\partial^2 \mathrm{w}}{\partial y^2}, \qquad c = -2z\frac{\partial^2 \mathrm{w}}{\partial x \partial y} \ldots (10).$$

The condition (c) is that the stress $R=0$. This enables us to obtain the component of strain g in terms of e and f. For isotropic matter with the constants λ and μ of I. art. 27, we have

$$(\lambda + 2\mu)\, g + \lambda (e+f) = 0.$$

The state of strain above described gives a state of stress of which the component stresses P, Q, R, U are known, viz. we have

$$\left.\begin{array}{ll} P = \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}(e+\sigma f), & Q = \dfrac{4\mu(\lambda+\mu)}{\lambda+2\mu}(\sigma e + f), \\ U = \mu c \qquad , & R = 0 \end{array}\right\} \ldots (11),$$

where σ is Poisson's ratio $\frac{1}{2}\lambda/(\lambda+\mu)$, and e, f, c are given by (9).

This state of stress gives rise to stress-couples

$$\left.\begin{array}{l} G_1 = -C\left(\dfrac{\partial^2 \mathrm{w}}{\partial x^2} + \sigma\dfrac{\partial^2 \mathrm{w}}{\partial y^2}\right), \\ G_2 = \quad C\left(\sigma\dfrac{\partial^2 \mathrm{w}}{\partial x^2} + \dfrac{\partial^2 \mathrm{w}}{\partial y^2}\right), \\ H = C(1-\sigma)\dfrac{\partial^2 \mathrm{w}}{\partial x \partial y} \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where C is the constant $\frac{8}{3}\mu h^3(\lambda+\mu)/(\lambda+2\mu)$. This constant will be called the *cylindrical rigidity*. Expressed in terms of the

Young's modulus E and the Poisson's ratio σ of the material we have

$$C = \tfrac{2}{3} Eh^3/(1-\sigma^2) \quad \text{......................(13)}.$$

Now we shall assume that when the plate is bent the state of strain above described is a first approximation to the state of strain that actually exists in any element of the plate; and from this assumption it follows that the above values of the stress-couples are, to a sufficient approximation, correct values. The assumption will be hereafter verified in the development of the general theory.

The general equation (9) for the flexure of a plate becomes

$$C\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) = 2h\left(Z + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) \quad \text{...... (14)}.$$

310. Transformation of Stress-resultants.

For the expression of the boundary-conditions it is necessary to obtain formulæ for the stress-resultants and stress-couples across any plane initially normal to the middle-surface. For this purpose we observe that the stress-resultants P_1, P_2,... being the integrals of the stress-components P,... with respect to z will be transformed by the same substitutions as the P,... while the stress-couples being the integrals with respect to z of the products of z and certain stress-components will be transformed by the same substitutions as these components. In fact G_1, G_2 and H are transformed by the same substitutions as P, $-Q$ and $-U$.

Consider a plane perpendicular to the middle-surface the normal to which drawn on the middle-surface at any point makes with the axis x an angle θ, and regard this normal as the x' axis of a new temporary system of coordinates (x', y', z') connected with the x, y, z system by the scheme

$$\left.\begin{array}{cccc} & x & y & z \\ x' & \cos\theta & \sin\theta & 0 \\ y' & -\sin\theta & \cos\theta & 0 \\ z' & 0 & 0 & 1 \end{array}\right\} .$$

To transform the stresses to the new system we may use the formulæ of I. art. 16; and, selecting those that we require, we have

$$\left.\begin{aligned} P' &= P\cos^2\theta + Q\sin^2\theta + U\sin 2\theta,\\ U' &= \tfrac{1}{2}\sin 2\theta\,(Q-P) + U\cos 2\theta,\\ T' &= S\sin\theta + T\cos\theta \end{aligned}\right\}\ldots\ldots\ldots\ldots(15).$$

These formulæ express the component stresses per unit area across the plane in question in terms of the component stresses per unit area across the planes x and y, viz. P', U', T' are the stresses per unit area of this plane in directions respectively normal to the plane, parallel to the trace of the plane on the middle-surface, and normal to the middle-surface.

The force- and couple-resultants across the same plane can now be written down. We take the components of the stress-resultant per unit length of the trace of the plane on the middle-surface to be P_1', U_1', T_1' in the directions of the normal to the plane, the trace of the plane on the middle-surface, and the normal to the middle-surface; also we take the components of the stress-couple per unit length of the same line to be H' and G_1' about the normal to the plane and the trace of the plane on the middle-surface; then we have

$$\left.\begin{aligned} P_1' &= P_1\cos^2\theta + P_2\sin^2\theta + U_1\sin 2\theta,\\ U_1' &= \tfrac{1}{2}\sin 2\theta\,(P_2-P_1) + U_1\cos 2\theta,\\ T_1' &= T_1\cos\theta + T_2\sin\theta,\\ G_1' &= G_1\cos^2\theta - G_2\sin^2\theta - H\sin 2\theta,\\ H' &= \tfrac{1}{2}\sin 2\theta\,(G_1+G_2) + H\cos 2\theta \end{aligned}\right\}\ldots\ldots\ldots(16).$$

These formulæ express the stress-resultants and stress-couples across the plane in question per unit length of the trace of this plane on the middle-surface, in terms of the stress-resultants and stress-couples across the planes perpendicular to the axes of x and y, per unit length of the traces of those planes on the middle-surface.

311. Boundary-Conditions[1].

The boundary-conditions express the statical equivalence of the system of forces directly applied to the edge, and the

[1] Thomson and Tait, *Nat. Phil.* Part II., art. 646.

$P_1', U_1', \ldots H'$ system arising from elastic reactions, the plane considered in the previous article being the plane normal to the middle-surface through the edge-line.

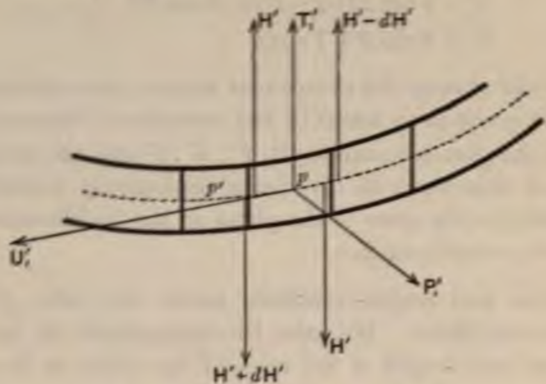


Fig. 52.

If p be any point on the edge-line, we may take p for the centre of an elementary rectangle whose dimensions are of the same order as the thickness, and reduce the system of forces acting on the rectangle to their force- and couple-resultants with p for origin.

The force-system has components per unit length of the edge-line which, in the notation of the last article, are P_1' normal to the plane, U_1' parallel to the edge-line, and T_1' in the plane and perpendicular to the edge-line.

The couple-system has components per unit length of the edge-line which are G_1' about the edge-line, and H' in the plane of the edge.

The system P_1', G_1' arising from forces normal to the edge admits of no further reduction.

The system U_1', T_1', H' consists entirely of forces acting in the plane of the rectangle. The couple $H'ds$ on any element may be regarded as consisting of two forces each H', acting at points on the edge-line distant ds and in direction normal to the middle-surface. If p, p' be the centres of two consecutive elementary rectangles, these couples are equivalent to forces $-dH'$ in the direction of the normal to the middle-surface at such points as the

point midway between p and p'. The distribution of couple H' is therefore equivalent to a distribution of force $-\dfrac{\partial H'}{\partial s}$ in the direction of the normal to the middle-surface. The system U_1', T_1', H' is therefore reducible to two forces U_1' along the edge-line, and $T_1' - \dfrac{dH'}{ds}$ in the direction of the normal to the middle-surface, each estimated per unit length of the edge-line.

Now let the forces $\mathfrak{P}$, $\mathfrak{U}$, $\mathfrak{T}$, and the couples $\mathfrak{G}$, $\mathfrak{H}$ per unit length of the edge-line be directly applied to the edge in the directions of the forces P_1', U_1', T_1', and about the axes of G_1' and H'. These will be in like manner equivalent to a tension $\mathfrak{P}$, a flexural couple $\mathfrak{G}$, a tangential force $\mathfrak{U}$ parallel to the edge-line, and a force $\mathfrak{T} - \dfrac{\partial \mathfrak{H}}{\partial s}$ normal to the middle-surface, each estimated per unit length of the edge-line.

The equivalence of the two systems is expressed by the equations

$$\left.\begin{array}{ll} P_1' = \mathfrak{P}, & U_1' = \mathfrak{U}, \\ G_1' = \mathfrak{G}, & T_1' - \dfrac{\partial H'}{\partial s} = \mathfrak{T} - \dfrac{\partial \mathfrak{H}}{\partial s} \end{array}\right\} \qquad \text{(17).}$$

According to the above if $\mathfrak{H}$ were increased by ξ, and $\mathfrak{T}$ diminished by $d\xi/ds$, no change would be produced in the boundary-conditions; and therefore such a system of forces being applied to the edge of a plate produces no sensible deformation. This result is a case of M. Boussinesq's theory of "local perturbations". We have just seen that such a system is an equilibrating system on each rectangular element of the boundary. When the plate has a very small finite thickness the application of such a system of forces will produce strains which are negligible at a very little distance from the edge of the plate.

The account just given of the boundary-conditions holds equally well whether the middle-surface is unextended or the extensions that lines of it undergo are of the order of strains in an elastic solid.

In the case of infinitesimal flexure unaccompanied by extension, which we are at present considering, the boundary-conditions are the last two of equations (17), and these are

$$-\cos^2\theta\, G_1 + \sin^2\theta\, G_2 + \sin 2\theta\, H = -\mathfrak{G},$$

and
$$-\sin\theta\left(\frac{\partial H}{\partial x} + \frac{\partial G_2}{\partial y}\right) + \cos\theta\left(\frac{\partial G_1}{\partial x} - \frac{\partial H}{\partial y}\right)$$
$$-\frac{\partial}{\partial s}\{\sin\theta\cos\theta\,(G_1 + G_2) + H\cos 2\theta\}$$
$$+ 2h\,(M\cos\theta - L\sin\theta) = \mathfrak{T} - \frac{\partial\mathfrak{H}}{\partial s}.$$

Expressed in terms of w these equations become

$$\left.\begin{aligned}
&C\left[\cos^2\theta\left(\frac{\partial^2 \mathrm{w}}{\partial x^2} + \sigma\frac{\partial^2 \mathrm{w}}{\partial y^2}\right)\right.\\
&\left.+\sin^2\theta\left(\sigma\frac{\partial^2 \mathrm{w}}{\partial x^2} + \frac{\partial^2 \mathrm{w}}{\partial y^2}\right) + \sin 2\theta\,(1-\sigma)\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right] = -\mathfrak{G},\\
&C\left[\sin\theta\frac{\partial}{\partial y}(\nabla^2\mathrm{w}) + \cos\theta\frac{\partial}{\partial x}(\nabla^2\mathrm{w})\right.\\
&\qquad\left.+(1-\sigma)\frac{\partial}{\partial s}\left\{\sin\theta\cos\theta\left(\frac{\partial^2 \mathrm{w}}{\partial y^2} - \frac{\partial^2 \mathrm{w}}{\partial x^2}\right) + \cos 2\theta\frac{\partial^2 \mathrm{w}}{\partial x\partial y}\right\}\right]\\
&\qquad\qquad - 2h\,(M\cos\theta - L\sin\theta) = -\left(\mathfrak{T} - \frac{\partial\mathfrak{H}}{\partial s}\right)
\end{aligned}\right\}\ldots(18).$$

If dn be the element of the normal to the edge-line drawn outwards, and ρ' the radius of curvature of this line, the above equations can be expressed in the forms

$$\left.\begin{aligned}
&C\left[\frac{\partial^2 \mathrm{w}}{\partial n^2} + \sigma\left(\frac{\partial^2 \mathrm{w}}{\partial s^2} + \frac{1}{\rho'}\frac{\partial \mathrm{w}}{\partial n}\right)\right] = -\mathfrak{G},\\
&C\left[\frac{\partial}{\partial n}(\nabla^2\mathrm{w}) + (1-\sigma)\frac{\partial}{\partial s}\left(\frac{\partial^2 \mathrm{w}}{\partial n\partial s}\right)\right]\\
&\qquad - 2h\,(M\cos\theta - L\sin\theta) = -\left(\mathfrak{T} - \frac{\partial\mathfrak{H}}{\partial s}\right)
\end{aligned}\right\}\ldots(19).$$

This transformation is given in Lord Rayleigh's *Theory of Sound*, vol. I., art. 216.

312. Transverse vibrations of Plates.

The equation of transverse vibration is obtained from the equation (14) by omitting the couples L and M, and replacing the force Z by $-\rho\dfrac{\partial^2 \mathrm{w}}{\partial t^2}$, where ρ is the density of the material of the plate. The differential equation is therefore

$$\frac{\partial^4 \mathrm{w}}{\partial x^4} + \frac{\partial^4 \mathrm{w}}{\partial y^4} + 2\frac{\partial^4 \mathrm{w}}{\partial x^2\partial y^2} = -\frac{2\rho h}{C}\frac{\partial^2 \mathrm{w}}{\partial t^2}\;\ldots\ldots\ldots\ldots(20).$$

When the plate is vibrating freely, the conditions that hold at the edge are (19) of the last article in which the right-hand members are put equal to zero. When the edge of the plate is simply supported the couple $\mathfrak{G}$ must vanish, but the other condition is that $w = 0$ at the edge; when the edge is built-in w and $\frac{\partial w}{\partial n}$ vanish at the edge.

In the case of a circular plate[1] the differential equation becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = -\frac{2\rho h}{C}\frac{\partial^2 w}{\partial t^2},$$

where r and θ are ordinary polar coordinates measured from the centre of the plate.

Taking w proportional to $e^{\iota pt}$, so that $p/2\pi$ is the frequency, the right-hand member becomes

$$\tfrac{3}{4}\frac{\rho p^2}{h^2\mu}\frac{\lambda + 2\mu}{\lambda + \mu} w = \kappa^4 w \text{ say},$$

and the differential equation becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \kappa^2\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \kappa^2\right) w = 0.$$

To satisfy this we take w to vary as $\cos(n\theta + \alpha)$, and then the general form for the displacement when any normal vibration is being executed is

$$w = [A_n J_n(\kappa r) + B_n J_n(\iota\kappa r)] \cos(n\theta + \alpha) e^{\iota pt} \ldots\ldots(21),$$

where A_n and B_n are constants, and J_n is Bessel's function of order n. If the plate be not complete up to the centre terms containing Bessel's functions of the second kind will have to be added.

For a circular plate of radius a vibrating freely the boundary-conditions are

$$\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) = 0,$$

$$\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + (1 - \sigma)\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right] = 0,$$

when $r = a$.

[1] The solution is due to Kirchhoff. (Crelle, XL., 1850.) The vibrations of elliptic plates have been discussed by Mathieu in Liouville's *Journal*, XIV., 1869, and by Barthélémy in the *Mémoires de l'Académie de Toulouse*, IX., 1877.

The ratio of $A_n : B_n$, giving the type, and the equation for the frequency are easily deduced. We find

$$-\frac{B_n}{A_n}=\frac{n^2(1-\sigma)\{\kappa a J_n'(\kappa a)-J_n(\kappa a)\}+\kappa^3a^3 J_n'(\kappa a)}{n^2(1-\sigma)\{\iota\kappa a J_n'(\iota\kappa a)-J_n(\iota\kappa a)\}-\iota\kappa^3a^3 J_n'(\iota\kappa a)}$$

$$=\frac{(1-\sigma)\{\kappa a J_n'(\kappa a)-n^2 J_n(\kappa a)\}+\kappa^2a^2 J_n(\kappa a)}{(1-\sigma)\{i\kappa a J_n'(\iota\kappa a)-n^2 J_n(\iota\kappa a)\}-\kappa^2a^2 J_n(\iota\kappa a)}.$$

For the discussion of the results, and the comparison with experiment, the reader is referred to Lord Rayleigh's *Theory of Sound*, vol. I. ch. X.

313. Equilibrium of Plates.

(α) Suppose a circular plate of radius a supports a load Z' per unit area symmetrically distributed[1].

The equation of equilibrium (14) becomes

$$C\left[\frac{\partial^4 \mathrm{w}}{\partial x^4}+\frac{\partial^4 \mathrm{w}}{\partial y^4}+2\,\frac{\partial^4 \mathrm{w}}{\partial x^2\partial y^2}\right]=Z' \quad\ldots\ldots\ldots\ldots\ldots (22),$$

and as the load is symmetrical, w will be a function of r the distance from the centre. Hence this equation may be replaced by

$$C\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathrm{w}}{dr}\right)\right\}\right]=Z'.$$

The complete primitive of this equation is

$$\mathrm{w}=\frac{1}{C}\int\frac{dr}{r}\int rdr\int\frac{dr}{r}\int rZ'dr+A+Br^2+A'\log r+B'r^2\log r.$$

As particular examples we may note the following[2]:—

(1°) A circular plate supported at its edge $r=a$ and strained by a uniform load. The conditions at the edge are

$$\mathrm{w}=0,\ \text{and}\ \frac{\partial^2\mathrm{w}}{\partial r^2}+\frac{\sigma}{r}\frac{\partial \mathrm{w}}{\partial r}=0\ \text{when}\ r=a,$$

and we find the deflexion at any point given by the equation

$$\mathrm{w}=\tfrac{1}{64}\frac{Z'}{C}(a^2-r^2)\left(\frac{5+2\sigma}{1+\sigma}\,a^2-r^2\right).$$

[1] Poisson, *Mém. Acad. Paris*, VIII., 1829, and Thomson and Tait, *Nat. Phil.* Part II, art. 649.

[2] The results only are stated and the work is left to the reader.

(2°) When the edge $r=a$ is built-in the deflexion is given by the equation

$$\mathrm{w}=\tfrac{1}{64}\frac{Z'}{C}(a^2-r^2)^2.$$

(β) In the case of an elliptic plate[1] whose boundary $(x^2/a^2+y^2/b^2=1)$ is built-in, the deflexion produced by uniform load Z' per unit area can be easily shewn to be

$$\mathrm{w}=\tfrac{1}{8}\frac{Z'}{C}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2\bigg/\left(\frac{3}{a^4}+\frac{3}{b^4}+\frac{2}{a^2b^2}\right),$$

and the deflexion produced by a load Z_1x can be easily shewn to be

$$\mathrm{w}=\tfrac{1}{24}\frac{Z_1x}{C}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^2\bigg/\left(\frac{5}{a^4}+\frac{1}{b^4}+\frac{2}{a^2b^2}\right).$$

We can hence write down the expression for the deflexion of an elliptic plate whose edge is built-in when immersed in liquid, for in this case the load per unit area can be expressed in the form $Z'+Z_1x+Z_2y$.

(γ) Taking the case of a rectangular plate[2] of sides a, b supported at the edges, and taking axes of x and y through one corner of the plate, the differential equation for the displacement produced by a load Z' per unit area is (22), and the boundary-conditions are that

$$\frac{\partial^2\mathrm{w}}{\partial x^2}+\sigma\frac{\partial^2\mathrm{w}}{\partial y^2}=0 \text{ when } x=0 \text{ or } x=a,$$

$$\frac{\partial^2\mathrm{w}}{\partial y^2}+\sigma\frac{\partial^2\mathrm{w}}{\partial x^2}=0 \text{ when } y=0 \text{ or } y=b,$$

and $\mathrm{w}=0$ at all points of the boundary.

The solution is of the form

$$\mathrm{w}=\frac{1}{\pi^4}\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}A_{mn}\frac{\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}}{\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^2},$$

where $$A_{mn}=\frac{4}{ab}\int_0^a\int_0^b\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\frac{Z'}{C}\,dx\,dy.$$

[1] I am indebted to Mr Bryan for these solutions.

[2] Saint-Venant's 'Annotated Clebsch', *note du* § 73.

(1°) For a uniform load

$$A_{mn} = \frac{16}{\pi^2 mn}\frac{Z'}{C}$$

if m and n be both odd. In other cases A_{mn} vanishes.

(2°) For an isolated load W at the centre of the rectangle

$$A_{mn} = \frac{4}{ab}\frac{W}{C}\sin\frac{m\pi}{2}\sin\frac{n\pi}{2},$$

and this vanishes unless m and n be both odd.

314. Examples on the Boundary-Conditions.

As further examples illustrating the application of the boundary-conditions we consider the following problems:

(1°) A rectangular plate bent into a circular cylinder whose generators are parallel to two edges of the plate.

We may take as the formula for the displacement

$$\text{w} = \tfrac{1}{2}\kappa y^2,$$

where κ is small; and then it is clear that at the edges $y = \text{const.}$, which become generators of the plate, we require flexural couples $G_2 = C\kappa$, and at the edges $x = \text{const.}$, which become circular sections, we require flexural couples $G_1 = -C\sigma\kappa$.

If the couples G_1 at the circular edges be not applied, there will be a tendency to anticlastic curvature of the middle-surface. The problem of a plate bent into a cylinder of finite curvature has been considered by Prof. Lamb. (See below, art. 355.)

(2°) A rectangular plate bent into the anticlastic surface

$$\text{w} = \tau xy,$$

where τ is small, and the axes of x and y are parallel to the edges[1].

The torsional couples $H, -H$ are constant, so that except at the corners no force need be applied to hold the plate. The forces at the corners reduce to two contrary pairs of equal forces $(1-\sigma)C\tau$ at the extremities of the two diagonals acting in the direction of the normal to the plate.

[1] Lamb, *Proc. Lond. Math. Soc.* XXI., 1890.

CHAPTER XX.

GENERAL THEORY OF THIN PLATES.

315. Preliminary. The Curvature of Surfaces.

Before commencing the general theory of elastic plates it is necessary to pay some attention to the theory of the curvature of surfaces. The deformation of a plate depends largely on the curvature of its middle-surface. The theory of curvature may be presented in two ways depending respectively on measurements made in space about the surface and on measurements made on the surface. The results obtained by the different methods explain and illustrate each other.

Taking first the ordinary theory of the indicatrix, principal curvatures, and lines of curvature, we observe that the equation of the part of a surface in the neighbourhood of any point, referred to the point as origin with the normal at the point as axis of z, and two rectangular lines in the tangent plane as axes of x and y, may be written

$$z = \tfrac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2 + 2\tau xy) \qquad \text{.................. (1)};$$

and the nature of the surface as regards curvature is completely expressed when the three quantities κ_1, κ_2, τ are known. The normals at two points near together do not in general intersect, but if we pass from a given point to a neighbouring point in one or other of two definite directions at right angles to each other the normals do intersect. These two directions are the principal tangents, or tangents to the lines of curvature, and, if they be chosen as axes of x and y, the term in xy will be missing from the equation (1). The quantities κ_1 and κ_2 are the curvatures of the

normal sections through the axes of x and y, and the quantity τ depends on the angle through which these axes must be turned in order to coincide with the principal tangents. In fact this angle is $\frac{1}{2}\tan^{-1} 2\tau/(\kappa_1 - \kappa_2)$.

The equation giving the two principal radii of curvature is

$$\rho^2(\kappa_1\kappa_2 - \tau^2) - \rho(\kappa_1 + \kappa_2) + 1 = 0 \qquad (2),$$

and the equation of the directions of the lines of curvature is

$$\tau(y^2 - x^2) - (\kappa_2 - \kappa_1)xy = 0 \qquad (3).$$

Thus the quantities κ_1, κ_2, τ are related in a definite way to the principal curvatures and the directions of the lines of curvature.

316. Gauss's method.

A different standpoint is taken in the theory propounded by Gauss, and we shall give an account of a modification of this theory. The surface is regarded as given, and we suppose two systems of curves traced upon it so as to form a network. The curves of the two systems are supposed to be distinguished by assigning different values to two parameters α and β, one for each system of curves. Then any point on the surface is determined by the values of the α and the β of the curves that pass through it. The length of the element of arc of the curve $\beta = \text{const.}$ between two curves α and $\alpha + d\alpha$ is some multiple of $d\alpha$, and we shall take it to be $Ad\alpha$. In like manner the element of the curve $\alpha = \text{const.}$ between the two curves β and $\beta + d\beta$ is taken to be $Bd\beta$. The quantities A and B are some functions of α and β. If the angle between the curves $\beta = \text{const.}$ and $\alpha = \text{const.}$ at the point (α, β) be χ, the square of the length ds of the line joining (α, β) to $(\alpha + d\alpha, \beta + d\beta)$ is given by the equation

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 + 2AB\,d\alpha\,d\beta\cos\chi \qquad (4),$$

in which A, B, and χ are supposed to be given functions of α and β.

Now suppose a system of moving axes of x, y, z to be constructed so that its origin is at the point (α, β), the axis z is the normal to the surface there, the axis x is the tangent to the line $\beta = \text{const.}$, and the axis y is in the tangent plane at right angles to the axis x. The axes constructed in the same way at $(\alpha + d\alpha, \beta + d\beta)$ will be obtained from those at α, β by translations $Ad\alpha + Bd\beta\cos\chi$, and $Bd\beta\sin\chi$ parallel to the axes of x and y at

(α, β), and by infinitesimal rotations about the axes x, y, z at (α, β) which can be expressed in the forms

$$p_1 d\alpha + p_2 d\beta, \quad q_1 d\alpha + q_2 d\beta, \quad r_1 d\alpha + r_2 d\beta \quad \ldots\ldots (5).$$

We may state the following relations connecting the quantities p, q, r, and A, B, χ:

$$\left.\begin{aligned}
\frac{\partial p_1}{\partial \beta} - \frac{\partial p_2}{\partial \alpha} &= q_1 r_2 - q_2 r_1, \\
\frac{\partial q_1}{\partial \beta} - \frac{\partial q_2}{\partial \alpha} &= r_1 p_2 - r_2 p_1, \\
\frac{\partial r_1}{\partial \beta} - \frac{\partial r_2}{\partial \alpha} &= p_1 q_2 - p_2 q_1, \\
r_1 &= -\frac{\partial \chi}{\partial \alpha} - \frac{1}{B \sin\chi}\left[\frac{\partial A}{\partial \beta} - \cos\chi \frac{\partial B}{\partial \alpha}\right], \\
r_2 &= \frac{1}{A \sin\chi}\left[\frac{\partial B}{\partial \alpha} - \cos\chi \frac{\partial A}{\partial \beta}\right], \\
-Aq_2 &= B\left[p_1 \sin\chi - q_1 \cos\chi\right]
\end{aligned}\right\} \quad \ldots\ldots\ldots (6).$$

The equation giving the principal radii of curvature is

$$\rho^2 (p_1 q_2 - p_2 q_1) + \rho\left[B(p_1 \cos\chi + q_1 \sin\chi) - Ap_2\right] + AB \sin\chi = 0 \quad \ldots\ldots\ldots (7),$$

and the differential equation of the lines of curvature is

$$B(p_2 \cos\chi + q_2 \sin\chi)\, d\beta^2 + Ap_1 d\alpha^2 + \left[B(p_1 \cos\chi + q_1 \sin\chi) + Ap_2\right] d\alpha d\beta = 0 \quad \ldots (8).$$

Proofs of all these formulæ are given by Darboux in his *Léçons sur la Théorie générale des Surfaces*, Paris, 1889. We shall give proofs in a note at the end of this volume.

317. Kinematics of Plates.

We consider an elastic plate as a very thin solid body bounded in its natural state by two plane faces very close together and by a cylindrical surface cutting them at right angles. It is supposed that the distance between the two planes (the thickness of the plate) is very small in comparison with any linear dimension of the curve cut out upon either of them by the cylindrical boundary. The plane midway between the two plane faces is called the *middle-surface*, the cylindrical boundary is called the *edge*, and the curve in which the middle-surface cuts the edge is called the *edge-line*. The strain of the plate depends largely on the deforma-

tion of the middle-surface, and this may be such that lines on the surface initially straight become finitely curved. In this case the plate is said to be finitely bent, and we must throughout the general theory contemplate this possibility.

The kinematical theory of the flexure of the plate is a geometrical theory of the curvature of its middle-surface after strain. We suppose that in the natural state two series of straight lines at right angles to each other are marked on the middle-surface so as to divide this surface into infinitesimal squares. Taking one corner of one square as origin, and the lines that meet there as axes of Cartesian rectangular coordinates (α, β), any point on the middle-surface will be given by its α and β. When the plate is strained so that the middle surface becomes a curved surface, the particle that was at a given point (α, β) on the unstrained middle-surface will occupy some position on the curved surface referred to, and we may regard α and β as parameters defining the position of a point on this surface. We suppose P to be any point of the middle-surface and that three rectangular line-elements (1, 2, 3) of the plate proceed from P, of which (1) is parallel to the axis α, (2) to the axis β, and (3) normal to the middle-surface. After strain these three line-elements do not remain rectangular, but by means of them we can construct a system of moving rectangular axes of x, y, z. We take the origin of this system at the strained position of P, the axis x along the line-element (1), the axis y in the tangent plane to the strained middle-surface and perpendicular to (1), and the axis z along the normal to the strained middle-surface. Then this is the same system of axes as that considered in the geometrical theory of § [illegible].

If the plate undergoes only ordinary strain the line-elements (1) and (2) will be very slightly extended and the angle between them will differ very little from a right angle. The initial lengths of these elements are $d\alpha$ and $d\beta$, and we shall suppose their lengths after strain to be $d\alpha(1+\epsilon_1)$ and $d\beta(1+\epsilon_2)$, so that ϵ_1 and ϵ_2 are the extensions of these lines. Also we shall suppose that ϖ is the cosine of the angle between them after strain, so that, with the notation of § [illegible],

$$\epsilon_1 = e_{xx}, \qquad \epsilon_2 = e_{yy}, \qquad \varpi = e_{xy} \qquad \ldots\ldots (1).$$

Kirchhoff, *Vorlesungen über Mathematische Physik, Mechanik*.

while χ differs very little from a right angle, and, if the square of ϖ be neglected, we have further

$$\chi = \tfrac{1}{2}\pi - \varpi \qquad \text{.............................(10).}$$

The quantity ϖ is the shear of the two rectangular line-elements (1) and (2).

The system of axes of x, y, z drawn at any point (α, β) of the strained middle-surface can be transformed into the corresponding system drawn at a neighbouring point $(\alpha + d\alpha, \beta + d\beta)$ by moving the origin through small distances $(1 + \epsilon_1)\, d\alpha + (1 + \epsilon_2)\, d\beta \cos \chi$, and $(1 + \epsilon_2)\, d\beta \sin \chi$ parallel to the axes of x and y, and by a rotation whose components about the axes of x, y, z will be taken to be $p_1 d\alpha + p_2 d\beta$, $q_1 d\alpha + q_2 d\beta$, $r_1 d\alpha + r_2 d\beta$. In the case of ordinary strains the translations must be reduced to $(1 + \epsilon_1)\, d\alpha + \varpi d\beta$, and $(1 + \epsilon_2)\, d\beta$.

With the values of A, B, χ in equations (9) and (10), the last three of equations (6) become

$$\left.\begin{array}{c} r_1 = \dfrac{\partial \varpi}{\partial \alpha} - \dfrac{\partial \epsilon_1}{\partial \beta}, \qquad r_2 = \dfrac{\partial \epsilon_2}{\partial \alpha}, \\ q_2 (1 + \epsilon_1) + p_1 (1 + \epsilon_2) - q_1 \varpi = 0 \end{array}\right\} \text{ .... (11);}$$

so that the quantities r_1, r_2, and $q_2 + p_1$ are small of the order of the extension of the middle-surface. The product $p_1 q_2 - p_2 q_1$ is, by the first of (6), small of the same order, and thus by (7) the measure of curvature is small of the same order. To a first approximation, when the extension is neglected, the middle-surface of the plate when finitely bent is a developable surface.

If now we suppose that some of the quantities $p_1, p_2, \ldots$ are finite we may reject the terms in ϵ_1, ϵ_2, ϖ, and write

$$r_1 = 0, \qquad r_2 = 0, \qquad q_2 = -p_1,$$

and then the equations (7) and (8) giving the principal radii of curvature and the lines of curvature become

$$\rho^2 (-p_2 q_1 - p_1^2) - \rho\, (p_2 - q_1) + 1 = 0,$$

and $$p_1 (d\beta^2 - d\alpha^2) - (p_2 + q_1)\, d\alpha d\beta = 0.$$

Comparing these with equations (2) and (3) we see that we have

$$\kappa_2 = p_2, \qquad \kappa_1 = -q_1, \qquad p_1 = \tau.$$

These are the equations connecting the Eulerian elements of curvature κ_1, κ_2, τ with the rotations of the two (p, q, r) systems. They are ultimately exact for finite bending accompanied by only infinitesimal extension.

Generally r_1, r_2, and $q_2 + p_1$ differ from zero, and $-q_1$, p_2, and p_1 differ from κ_1, κ_2, and τ by quantities of the order of the extension of the middle-surface.

318. Kinematical Equations.

We give here the analysis by which the quantities p_1, p_2, q_1, q_2, r_1, r_2 are determined. For this purpose we have to assume a system of fixed axes of ξ, η, ζ, and suppose them connected with the moving axes of x, y, z by the scheme of 9 direction-cosines

$$\left.\begin{array}{cccc} & \xi, & \eta, & \zeta \\ x, & l_1, & m_1, & n_1 \\ y, & l_2, & m_2, & n_2 \\ z, & l_3, & m_3, & n_3 \end{array}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

The rotations executed by the system of axes of x, y, z about themselves as we pass from the point (α, β) to the point $(\alpha + d\alpha, \beta + d\beta)$ are $p_1 d\alpha + p_2 d\beta \ldots$, where

$$\left.\begin{aligned} p_1 d\alpha + p_2 d\beta &= l_3 dl_2 + m_3 dm_2 + n_3 dn_2, \\ q_1 d\alpha + q_2 d\beta &= l_1 dl_3 + m_1 dm_3 + n_1 dn_3, \\ r_1 d\alpha + r_2 d\beta &= l_2 dl_1 + m_2 dm_1 + n_2 dn_1 \end{aligned}\right\} \quad\ldots\ldots\ldots (13),$$

in which any differential as dl_1 means $(\partial l_1/\partial\alpha)\, d\alpha + (\partial l_1/\partial\beta)\, d\beta$.

Since the ratio $d\alpha : d\beta$ may be any real number we find

$$\left.\begin{aligned} p_1 &= l_3 \frac{\partial l_2}{\partial\alpha} + m_3 \frac{\partial m_2}{\partial\alpha} + n_3 \frac{\partial n_2}{\partial\alpha}, \\ q_1 &= l_1 \frac{\partial l_3}{\partial\alpha} + m_1 \frac{\partial m_3}{\partial\alpha} + n_1 \frac{\partial n_3}{\partial\alpha}, \\ r_1 &= l_2 \frac{\partial l_1}{\partial\alpha} + m_2 \frac{\partial m_1}{\partial\alpha} + n_2 \frac{\partial n_1}{\partial\alpha} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (14),$$

and p_2, q_2, r_2 are obtained from these by writing β for α.

It is to be noticed that l_2, m_2, n_2 are not the direction-cosines of the line-element (2) after strain. These direction-cosines are l_2', m_2', n_2', where

$$l_2' = l_2 + l_1\varpi, \qquad m_2' = m_2 + m_1\varpi, \qquad n_2' = n_2 + n_1\varpi \ldots (15)$$

to the first order in ϖ.

Now if ξ, η, ζ be the coordinates of a point P of the middle-surface after strain, and $\xi+d\xi$, $\eta+d\eta$, $\zeta+d\zeta$ those of a neighbouring point P', we have

$$\left.\begin{array}{lll} l_1(1+\epsilon_1)=\dfrac{\partial\xi}{\partial\alpha}, & m_1(1+\epsilon_1)=\dfrac{\partial\eta}{\partial\alpha}, & n_1(1+\epsilon_1)=\dfrac{\partial\zeta}{\partial\alpha}, \\ l_2'(1+\epsilon_2)=\dfrac{\partial\xi}{\partial\beta}, & m_2'(1+\epsilon_2)=\dfrac{\partial\eta}{\partial\beta}, & n_2'(1+\epsilon_2)=\dfrac{\partial\zeta}{\partial\beta} \end{array}\right\}\quad\ldots(16).$$

Also the direction-cosines l_3, m_3, n_3 of the normal to the middle-surface after strain are given by such equations as

$$l_3=m_1n_2-m_2n_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

The formulæ given in this article enable us to determine the p's, q's, and r's in terms of differential coefficients of ξ, η, ζ with respect to α, β.

319. General Theory of Plates.

Entering now upon Gehring's theory of thin elastic plates, we revert in the first place to the principle explained in art. 249, according to which the plate may be regarded as made up of very small prisms each of which has all its linear dimensions of the same order of magnitude as the thickness of the plate. The general elastic equations apply not to the plate but to one of these prisms, and the strain in such a prism may be investigated on the supposition that no bodily forces act upon it. We shall apply the principles explained in art. 249 to find the stress-couples that act upon an element. The steps of the process are as follows:—We first investigate certain differential identities connecting the displacements of any point in a prism with the quantities that define the extension and curvature of the middle-surface. We next, in accordance with a method of approximation to be explained hereafter, reduce these identities to a simpler form. From the differential equations so obtained, values of the strain-components and corresponding stresses can be deduced involving arbitrary functions. The arbitrary functions are determined by means of the equations of equilibrium and the boundary-conditions at the surfaces of the element that initially coincided with the plane faces of the plate. The stresses are thus found approximately, and the stress-couples are easily deduced.

As there has been so much controversy on the subject it may be as well to point out which parts of the theory are fully established. The differential identities in their first form are rigorously proved. The approximate equations that replace them have no pretence to rigour, and they have been objected to as not being sufficiently approximate[1]. They carry with them the remainder of the theory including the expressions for the stress-couples. It is noteworthy that the writers who reject them obtain the same results including the values of these couples. It is hoped that the method of approximation to be given presently will tend to remove the objection.

320. Differential Identities.

In the notation of art. 317 we suppose P to be a point on the middle-surface of the plate, and take P for the centre of an elementary prism whose dimensions are all of the same order of magnitude as the thickness. The boundaries of the prism are the two plane surfaces of the plate and pairs of planes perpendicular to the axes of α and β drawn as in art. 317.

Let Q be the position of any particle of the plate near to P; and let the coordinates of Q referred to the line-elements at P before strain be x, y, z; after strain, referred to the axes of x, y, z let them be $x+u$, $y+v$, $z+w$. Then u, v, w are the displacements of Q, and if α, β be the coordinates of P before strain referred to the axes of α, β, u, v, and w are functions of x, y, z, α and β.

Suppose P' is a point on the middle-surface near to P, then we might refer Q to P' instead of P. The coordinates of Q before strain referred to P' are $x-d\alpha$, $y-d\beta$, z, where $d\alpha$ and $d\beta$ are the coordinates of P' referred to P before strain. To obtain the values of the displacements of Q referred to P' we must in the expressions for u, v, w replace α and β by $\alpha+d\alpha$ and $\beta+d\beta$, and replace x and y by $x-d\alpha$ and $y-d\beta$. Hence the coordinates of Q referred to P' after strain are

$$\left.\begin{aligned} &x-d\alpha+u+\frac{\partial u}{\partial\alpha}d\alpha+\frac{\partial u}{\partial\beta}d\beta-\frac{\partial u}{\partial x}d\alpha-\frac{\partial u}{\partial y}d\beta,\\ &y-d\beta+v+\frac{\partial v}{\partial\alpha}d\alpha+\frac{\partial v}{\partial\beta}d\beta-\frac{\partial v}{\partial x}d\alpha-\frac{\partial v}{\partial y}d\beta,\\ &z\qquad\;\;+w+\frac{\partial w}{\partial\alpha}d\alpha+\frac{\partial w}{\partial\beta}d\beta-\frac{\partial w}{\partial x}d\alpha-\frac{\partial w}{\partial y}d\beta \end{aligned}\right\}\text{......(18).}$$

[1] Cf. p. 93, footnote (2).

Now since the axes of x, y, z at P' after strain are obtained from those at P by translations $(1+\epsilon_1)\,d\alpha+\varpi d\beta$, $(1+\epsilon_2)\,d\beta$ parallel to the axes of x and y, and by rotations

$$p_1 d\alpha + p_2 d\beta, \quad q_1 d\alpha + q_2 d\beta, \quad r_1 d\alpha + r_2 d\beta$$

about the axes of x, y, and z, the coordinates of Q referred to P' after strain are

$$\left.\begin{aligned} &x+u+(y+v)(r_1 d\alpha+r_2 d\beta)-(z+w)(q_1 d\alpha+q_2 d\beta)\\ &\qquad\qquad\qquad\qquad -(1+\epsilon_1)\,d\alpha-\varpi d\beta,\\ &y+v+(z+w)(p_1 d\alpha+p_2 d\beta)-(x+u)(r_1 d\alpha+r_2 d\beta)\\ &\qquad\qquad\qquad\qquad -(1+\epsilon_2)\,d\beta,\\ &z+w+(x+u)(q_1 d\alpha+q_2 d\beta)-(y+v)(p_1 d\alpha+p_2 d\beta) \end{aligned}\right\}\ldots(19).$$

Comparing the two expressions (18) and (19) for the coordinates of Q referred to P', and observing that the ratio $d\alpha : d\beta$ may be any real number, we see that we may equate coefficients of $d\alpha$ and $d\beta$, and obtain six equations, viz.:

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \alpha} - r_1(y+v) + q_1(z+w) + \epsilon_1,\\ \frac{\partial v}{\partial x} &= \frac{\partial v}{\partial \alpha} - p_1(z+w) + r_1(x+u),\\ \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial \alpha} - q_1(x+u) + p_1(y+v) \end{aligned}\right\}\ldots\ldots\ldots(20),$$

and

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \beta} - r_2(y+v) + q_2(z+w) + \varpi,\\ \frac{\partial v}{\partial y} &= \frac{\partial v}{\partial \beta} - p_2(z+w) + r_2(x+u) + \epsilon_2,\\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial \beta} - q_2(x+u) + p_2(y+v) \end{aligned}\right\}\ldots\ldots\ldots(21).$$

321. Method of Approximation.

We proceed to indicate a method of successive approximation whereby forms for the displacements u, v, w can be determined from the identities (20) and (21). The method depends on the facts that x, y, z are everywhere small, and u, v, w are small in comparison with x, y, z, while ϵ_1, ϵ_2, ϖ are small of the order of strains in an elastic solid at most.

There are three classes of cases to be considered. In the first some at least of the quantities $p_1, p_2, \ldots$ defining the curvature

may be finite, or the plate is finitely bent. In this case we may reject the terms in ϵ_1, ϵ_2, ϖ as small compared with terms of the order pz, where p is one of the p, q, r which is finite. There will exist modes of infinitesimal deformation continuous with these for which the p's are infinitesimal, while ϵ_1, ϵ_2, ϖ are so small as to be negligible in comparison with terms of the order $p \times$ (thickness of plate). In the second class of cases quantities of the order pz are negligible in comparison with ϵ_1, ϵ_2, ϖ. These correspond to simple extension of the plate without flexure. In the third class of cases quantities of the order pz are comparable with quantities of the order ϵ_1. It is clear that if we retain in all cases quantities of the order pz and quantities of the order ϵ_1 we shall be at liberty to reject at the end those terms which may be small in consequence of the case under discussion falling into the first or second class.

Now we may suppose that u, v, w are capable of expansion in integral powers of x, y, z with coefficients some functions of α, β. Since x, y, z are everywhere small, quantities such as $\partial u/\partial x$ are great compared with quantities of the order u, while quantities such as $\partial u/\partial \alpha$ are of the same order as u[1]. We may therefore for a first approximation reject the differential coefficients of u, v, w with respect to α and β. But without making the supposition that u, v, w are capable of such expansion we can still see that these terms are to be rejected. For $\partial u/\partial \alpha$ is the limit of a fraction whose denominator is the distance between the centres of contiguous elementary prisms which lie on a line $\beta =$ const., and whose numerator is the difference of the values of the displacements of homologous points of these prisms referred to their centres; and this fraction must be small in comparison with $\partial u/\partial x$; for the latter is the limit of a fraction whose denominator is the distance between two points of the same prism, and whose numerator is the difference of the values of the displacements of these points referred to the centre of the same prism.

Since u, v, w are small compared with x, y, z, such terms as $p_1 w$ are to be rejected in comparison with such terms as $p_1 z$.

The equations (20) and (21) are reduced by the omission of such terms as $\partial u/\partial \alpha$ and such terms as $p_1 w$ to the forms

[1] Really $u \div$ (the unit of length).

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=-r_1y+q_1z+\epsilon_1, & \frac{\partial u}{\partial y}&=-r_2y+q_2z+\varpi,\\ \frac{\partial v}{\partial x}&=-p_1z+r_1x, & \frac{\partial v}{\partial y}&=-p_2z+r_2x+\epsilon_2,\\ \frac{\partial w}{\partial x}&=-q_1x+p_1y, & \frac{\partial w}{\partial y}&=-q_2x+p_2y\end{aligned}\right\}\ldots(22).$$

These equations are incompatible unless

$$r_1=r_2=0, \text{ and } p_1+q_2=0.$$

Now we have already seen in art. 317 that r_1, r_2, and p_1+q_2 are quantities of the same order of magnitude as ϵ_1, ϵ_2, ϖ,[1] and, in the first part of the present article, that any terms known to be of the order $\epsilon_1 z$ are to be rejected. It follows that we are at liberty still further to simplify our equations by omitting terms in r_1 and r_2 and putting $-p_1$ for q_2. Also we have seen in art. 317 that the quantities $-q_1$, p_2, and p_1 differ from the Eulerian elements of curvature κ_1, κ_2, τ by quantities of the same order of magnitude as ϵ_1, ϵ_2, ϖ, and we may therefore in equations (22) write τ for p_1, κ_2 for p_2, $-\kappa_1$ for q_1, and $-\tau$ for q_2.

Suppose the approximate equations that replace (20) and (21) written down in accordance with the principles just explained, and suppose that a solution of them has been found involving arbitrary functions. The arbitrary functions can be determined by means of the general elastic equations and the boundary-conditions at the faces of the plate, and the general equations may for this purpose be simplified by omitting bodily forces and kinetic reactions since we are dealing with an elementary prism, (see art. 249.) The results obtained constitute a first approximation to the displacements, strains, and stresses in such a prism. A second approximation can then be found by substituting in equations (20) and (21) the values found in the first approximation and solving again. Arbitrary functions will have again to be introduced, and might be found as before by recourse to the differential equations and the boundary-conditions, and it will be necessary in this second approximation to retain the bodily forces and kinetic reactions. The process might be continued indefinitely but it is quite unnecessary to carry out more than the first approximation. The formulation of the method is however valuable.

[1] Really $\epsilon_1 \div$ (the unit of length), ...

322. First Approximation. *Internal Strain.*

According to the method explained we can now reduce equations (20) and (21) to

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= -\kappa_1 z + \epsilon_1, & \frac{\partial u}{\partial y} &= -\tau z + \varpi,\\ \frac{\partial v}{\partial x} &= -\tau z, & \frac{\partial v}{\partial y} &= -\kappa_2 z + \epsilon_2,\\ \frac{\partial w}{\partial x} &= \kappa_1 x + \tau y, & \frac{\partial w}{\partial y} &= \tau x + \kappa_2 y\end{aligned}\right\} \qquad (23)$$

by omitting terms containing such quantities as u, $\frac{\partial u}{\partial \alpha}$, and replacing the p, q, r system by the κ_1, κ_2, τ system.

Integrating these equations we find

$$\left.\begin{aligned}u &= u_0 - \kappa_1 zx - \tau zy + \epsilon_1 x + \varpi y,\\ v &= v_0 - \tau zx - \kappa_2 zy + \epsilon_2 y,\\ w &= w_0 + \tfrac{1}{2}(\kappa_1 x^2 + \kappa_2 y^2 + 2\tau xy)\end{aligned}\right\} \qquad (24),$$

in which u_0, v_0, w_0 are functions of z.

From these we deduce the strain-components in the forms

$$\left.\begin{aligned}e &= \epsilon_1 - \kappa_1 z, & a &= \frac{\partial v_0}{\partial z},\\ f &= \epsilon_2 - \kappa_2 z, & b &= \frac{\partial u_0}{\partial z},\\ g &= \frac{\partial w_0}{\partial z}, & c &= \varpi - 2\tau z\end{aligned}\right\} \qquad (25),$$

and we notice that, to the order of approximation to which the work has been carried, the strains, and therefore also the stresses, are independent of x and y. The order of approximation in question is such that the strains are of the first order in the small quantities x, y, z.

The equations of equilibrium become

$$\frac{\partial T}{\partial z} = 0, \qquad \frac{\partial S}{\partial z} = 0, \qquad \frac{\partial R}{\partial z} = 0 \qquad (26),$$

in which bodily forces and kinetic reactions are omitted for the reason explained in art. 249.

The boundary-conditions that hold at the surfaces which were initially plane faces of the plate, when *no external surface-tractions are applied* to these faces, are

$$T=0,\qquad S=0,\qquad R=0 \text{ ....(27).}$$

From the equations (26) and the conditions (27) it follows that, to the order of approximation to which the work has been carried, the stresses R, S, T vanish at all points of any elementary prism of the plate.

When external surface-tractions are applied to the initially plane faces, the same result may be assumed to hold good. The tractions in question are to be replaced by bodily forces and couples acting directly upon the element of volume of the plate bounded by two pairs of normal sections and the plane faces, and the equations of equilibrium formed in accordance with the principle enunciated in art. 306.

It is important to observe that if a further approximation were made the stresses R, S, T would not vanish, and although, as we shall see, it is unnecessary to make the second approximation the stresses S and T nevertheless play a very important part.

323. First Approximation to the Stress-components.

Supposing the material of the plate isotropic, we have the expressions of the stresses in terms of the strains by means of the equations

$$P=(\lambda+2\mu)\,e+\lambda\,(f+g),\ \dots\qquad S=\mu a,\ \dots$$

Since R, S, T all vanish we find equations (25) taking the form

$$\left.\begin{aligned} e&=\epsilon_1-\kappa_1 z, & a&=0,\\ f&=\epsilon_2-\kappa_2 z, & b&=0,\\ g&=-\frac{\lambda}{\lambda+2\mu}\{\epsilon_1+\epsilon_2-(\kappa_1+\kappa_2)\,z\}, & c&=\varpi-2\tau z \end{aligned}\right\}\text{...(28).}$$

This gives the six strains as far as terms in z.

In like manner as far as quadratic terms in x, y, z the displacements are

$$\left.\begin{aligned} u&=-\kappa_1 zx-\tau zy+\epsilon_1 x+\varpi y,\\ v&=-\tau zx-\kappa_2 zy+\epsilon_2 y.\\ w&=-\frac{\lambda}{\lambda+2\mu}\{(\epsilon_1+\epsilon_2)\,z-\tfrac{1}{2}(\kappa_1+\kappa_2)\,z^2\}+\tfrac{1}{2}(\kappa_1 x^2+\kappa_2 y^2+2\tau xy) \end{aligned}\right\}\text{(29).}$$

As far as terms in z the stresses are

$$\left.\begin{array}{ll} P=\dfrac{3C}{2h^3}\{\epsilon_1+\sigma\epsilon_2-(\kappa_1+\sigma\kappa_2)z\}, & R=0, \\ Q=\dfrac{3C}{2h^3}\{\epsilon_2+\sigma\epsilon_1-(\kappa_2+\sigma\kappa_1)z\}, & S=0, \\ U=\dfrac{3C(1-\sigma)}{4h^3}(\varpi-2\tau z), & T=0 \end{array}\right\}\text{......(30)},$$

where $\qquad C=\frac{8}{3}\mu h^3(\lambda+\mu)/(\lambda+2\mu)$ as in art. 309,

and $\qquad \sigma=\frac{1}{2}\lambda/(\lambda+\mu)$.

With these values the potential energy per unit volume, of which the general expression is $\frac{1}{2}(Pe+Qf+Rg+Sa+Tb+Uc)$, becomes

$$\frac{3C}{4h^3}[(\epsilon_1-\kappa_1 z)^2+(\epsilon_2-\kappa_2 z)^2+2\sigma(\epsilon_1-\kappa_1 z)(\epsilon_2-\kappa_2 z)+\tfrac{1}{2}(1-\sigma)(\varpi-2\tau z)^2].$$

Multiplying this expression by dz, and integrating between the limits $-h$ and h for z, we have the potential energy per unit area in the form

$$\tfrac{1}{2}C[(\kappa_1+\kappa_2)^2-2(1-\sigma)(\kappa_1\kappa_2-\tau^2)]$$
$$+\tfrac{3}{2}\frac{C}{h^2}[(\epsilon_1+\epsilon_2)^2-2(1-\sigma)(\epsilon_1\epsilon_2-\tfrac{1}{4}\varpi^2)]\text{............(31)}.$$

The first line of this is the potential energy due to bending, and the second line is that due to stretching of the middle-surface.

324. Stress-resultants and Stress-couples.

For the purpose of forming the equations of equilibrium or small motion of an element of the plate bounded by normal sections through two pairs of lines $\alpha=$ const. and $\beta=$ const. near together we have to reduce the stresses on the faces of such an element to force- and couple-resultants. These will be estimated per unit length of the lines in which the faces in question cut the middle-surface.

On the element of the face $\alpha=$ const. whose trace on the middle-surface is $d\beta$ there act the forces of the P, U, T system. These reduce to a force at (α, β) whose components parallel to the axes of x, y, z will be taken to be

$$P_1 d\beta, \qquad U_1 d\beta, \qquad T_1 d\beta$$

respectively, and a couple whose components about the axes of x and y will be taken to be

$$H_1 d\beta, \qquad G_1 d\beta.$$

In like manner, on the element of the face $\beta = \text{const.}$ whose trace on the middle-surface is $d\alpha$ there act the forces of the U, Q, S system. These reduce to a force at (α, β) whose components parallel to the axes of x, y, z will be taken to be

$$U_2 d\alpha, \qquad P_2 d\alpha, \qquad T_2 d\alpha$$

respectively, and a couple whose components about the axes of x and y will be taken to be

$$G_2 d\alpha, \qquad H_2 d\alpha.$$

The quantities P_1, P_2, U_1, U_2, T_1, T_2 will be called *stress-resultants* and the quantities G_1, G_2, H_1, and H_2, *stress-couples.* P_1 and P_2 are tensions per unit length of the traces on the middle-surface of the faces across which they act; U_1 and U_2 are tangential stresses in the middle-surface; and T_1 and T_2 tangential stresses directed along the normal to the middle-surface. G_1 and G_2 are flexural couples about the traces on the middle-surface of the faces across which they act; H_1 and H_2 are torsional couples in the faces across which they act.

The stress-resultants are given by the equations

$$\left.\begin{aligned} &P_1 = \int_{-h}^{h} P dz, \qquad P_2 = \int_{-h}^{h} Q dz, \qquad U_1 = U_2 = \int_{-h}^{h} U dz, \\ &T_1 = \int_{-h}^{h} T dz, \qquad T_2 = \int_{-h}^{h} S dz \end{aligned}\right\} \ldots (32).$$

The stress-couples are given the equations

$$\left.\begin{aligned} &G_1 = \int_{-h}^{h} Pz dz, \qquad G_2 = \int_{-h}^{h} -Qz dz, \\ &H_1 = -H_2 = \int_{-h}^{h} -Uz dz = H \text{ say} \end{aligned}\right\} \ldots\ldots\ldots (33).$$

With the approximate values of the stresses given in (30) we hence find

$$\left.\begin{aligned} &P_1 = \frac{3C}{h^2}(\epsilon_1 + \sigma\epsilon_2), \quad P_2 = \frac{3C}{h^2}(\epsilon_2 + \sigma\epsilon_1), \quad U_1 = U_2 = \frac{3C(1-\sigma)}{2h^2}\varpi, \\ &T_1 = T_2 = 0, \\ &G_1 = -C(\kappa_1 + \sigma\kappa_2), \quad G_2 = C(\kappa_2 + \sigma\kappa_1), \quad H = C(1-\sigma)\tau \end{aligned}\right\} (34).$$

It is important to notice that the values given in (34) for $-G_1$, G_2, H are the partial differential coefficients with respect to κ_1, κ_2, τ of the first line of the expression (31) for the potential energy, and the values given for P_1, P_2, U_1 are the partial differential coefficients of the second line of the same expression with respect to ϵ_1, ϵ_2, ϖ.

It will be seen below that the values of P_1, P_2, U_1, U_2, and T_1, T_2, apply only to the case of extension of the plate, and the values of G_1, G_2, and H only to the case of flexure.

325. Effect of Second Approximation.

To the order of approximation to which the work has been carried there are no resultant stresses parallel to the axis z. The order in question is such that the expressions for the elastic stresses which give rise to the resultants contain only first powers of the coordinates.

If a second approximation were made the stresses P, Q, R, S, T, U might differ from the expressions given for them in equations (30) by terms containing the second and higher powers of the coordinates x, y, z, or the thickness h. These would give rise to stress-resultants which, so far as h is concerned, would be of the third or some higher order in h. These terms may be of any order of magnitude in comparison with those found in (34), and they will have to be retained or rejected according as the case under discussion falls into one or another of the classes described in art. 321. The corrections to the values of the stress-couples arising from the corrections to the values of the stresses will be of the fourth or a higher order in h and they may in all cases be disregarded.

In the first and third classes of cases, *i.e.* whenever there is flexure, the stress-couples are determined in terms of it by the formulæ of the last article, but the stress-resultants are not sufficiently determined. In the second class of cases, *i.e.* when there is extension and no flexure, the stress-resultants are determined in terms of the extension by the formulæ (34) of the last article, and the stress-couples are unimportant.

When there is flexure it is unnecessary to proceed to a second approximation in order to determine the stress-resultants. They

may be introduced as unknowns into the equations of equilibrium or small motion of the plate and determined or rather eliminated by a direct process.

326. General Equations of Equilibrium. *Small displacements*[1].

Now suppose a thin plane plate is slightly deformed and held in its new position by applied forces and couples. Let a system of moving axes of x, y, z be constructed as in art. 317. The displacements being supposed so small that their squares may be neglected, we shall arrive at sufficiently exact results if we neglect the changes of direction of these axes that accompany changes of their origin (α, β). Consider the equilibrium of an element of the plate contained by the naturally plane faces and by the four normal sections which in the natural state cut out on the middle-surface a rectangle bounded by the lines α, β, $\alpha + d\alpha$, and $\beta + d\beta$. Let the external forces applied to this element be reduced to a force at the corner (α, β), whose components parallel to the axes of x, y, z drawn through that point are $2Xhd\alpha d\beta$, $2Yhd\alpha d\beta$, $2Zhd\alpha d\beta$, and a couple, whose moments about the axes of x and y are $2Lhd\alpha d\beta$, and $2Mhd\alpha d\beta$. Let the part of the plate on the side of the normal section through the bounding line $\alpha = \text{const.}$ remote from the element act upon the part on the other side with a resultant force and couple such that the components of the force per unit length of the curve $\alpha = \text{const.}$ at the point (α, β) are $-P_1, -U_1, -T_1$ parallel to the axes of x, y, z, and the components of the couple per unit length of the same line are $-H$ and $-G_1$ about the axes of x and y. Also let the part of the plate on the side of the bounding curve $\beta = \text{const.}$ remote from the element act upon the part on the other side with a resultant force and couple such that the components of the force per unit length of the curve $\beta = \text{const.}$ at the point (α, β) are $-U_1, -P_2, -T_2$ parallel to the axes of x, y, z, and the components of the couple per unit length of the same line are $-G_2$ and H about the axes of x and y.

The forces $P_1, P_2, \dots$ are the stress-resultants, and the couples G_1, G_2, H are the stress-couples per unit length of the bounding lines of the section of the element by the middle-surface of the plate. In the case of pure extension the couples vanish, and we

[1] The equations for a plate finitely bent will be given in art. 341.

know a sufficient approximation to the stress-resultants. When there is flexure we know a sufficient approximation to the stress-couples, but the stress-resultants are unknown.

Exactly as in art. 308 we can obtain the equations of equilibrium in the form:—three equations of resolution

$$\left.\begin{aligned} \frac{\partial P_1}{\partial \alpha} + \frac{\partial U_1}{\partial \beta} + 2hX = 0, \\ \frac{\partial U_1}{\partial \alpha} + \frac{\partial P_2}{\partial \beta} + 2hY = 0, \\ \frac{\partial T_1}{\partial \alpha} + \frac{\partial T_2}{\partial \beta} + 2hZ = 0 \end{aligned}\right\} \qquad (35);$$

and two equations of moments

$$\left.\begin{aligned} \frac{\partial H}{\partial \alpha} + \frac{\partial G_2}{\partial \beta} + T_2 + 2hL = 0, \\ \frac{\partial G_1}{\partial \alpha} - \frac{\partial H}{\partial \beta} - T_1 + 2hM = 0 \end{aligned}\right\} \qquad (36).$$

The boundary-conditions have already been given in art. 311, equations (17).

327. Flexure and Extension of plate.

The equations that determine the small displacements of the plate can be found by using the analysis developed in art. 318. It is easy to shew that if u, v, w be the displacements parallel to the axes of α and β and normal to the middle-surface the extension is defined by the equations

$$\epsilon_1 = \frac{\partial u}{\partial \alpha}, \qquad \epsilon_2 = \frac{\partial v}{\partial \beta}, \qquad \varpi = \frac{\partial v}{\partial \alpha} + \frac{\partial u}{\partial \beta},$$

and the curvature of the strained middle-surface is defined by the equations

$$\kappa_1 = \frac{\partial^2 w}{\partial \alpha^2}, \qquad \kappa_2 = \frac{\partial^2 w}{\partial \beta^2}, \qquad \tau = \frac{\partial^2 w}{\partial \alpha \partial \beta}.$$

When there is flexure unaccompanied by extension u and v vanish and we have the theory of ch. xix.

When there is extension unaccompanied by flexure, w vanishes, and the equations to determine u and v are the first two of (35), in which P_1, P_2, U_1 are given in terms of ϵ_1, ϵ_2, ϖ by the first three of (34).

As an example we may consider the equations of extensional vibration. Replacing X and Y by $-\rho\partial^2\mathrm{u}/\partial t^2$, and $-\rho\partial^2\mathrm{v}/\partial t^2$, where ρ is the density of the material of the plate, the equations are

$$\frac{3C}{h^2}\left[\frac{\partial}{\partial\alpha}\left(\frac{\partial\mathrm{u}}{\partial\alpha}+\sigma\frac{\partial\mathrm{v}}{\partial\beta}\right)+\tfrac{1}{2}(1-\sigma)\frac{\partial}{\partial\beta}\left(\frac{\partial\mathrm{v}}{\partial\alpha}+\frac{\partial\mathrm{u}}{\partial\beta}\right)\right]=2\rho h\frac{\partial^2\mathrm{u}}{\partial t^2},$$

$$\frac{3C}{h^2}\left[\frac{\partial}{\partial\beta}\left(\frac{\partial\mathrm{v}}{\partial\beta}+\sigma\frac{\partial\mathrm{u}}{\partial\alpha}\right)+\tfrac{1}{2}(1-\sigma)\frac{\partial}{\partial\alpha}\left(\frac{\partial\mathrm{v}}{\partial\alpha}+\frac{\partial\mathrm{u}}{\partial\beta}\right)\right]=2\rho h\frac{\partial^2\mathrm{v}}{\partial t^2}.$$

The boundary-conditions at a free edge whose normal makes an angle θ with the axis α are

$$\left(\frac{\partial\mathrm{u}}{\partial\alpha}+\sigma\frac{\partial\mathrm{v}}{\partial\beta}\right)\cos^2\theta+\left(\frac{\partial\mathrm{v}}{\partial\beta}+\sigma\frac{\partial\mathrm{u}}{\partial\alpha}\right)\sin^2\theta+\tfrac{1}{2}(1-\sigma)\left(\frac{\partial\mathrm{v}}{\partial\alpha}+\frac{\partial\mathrm{u}}{\partial\beta}\right)\sin 2\theta=0,$$

$$\left(\frac{\partial\mathrm{v}}{\partial\beta}+\frac{\partial\mathrm{u}}{\partial\alpha}\right)\sin 2\theta+\left(\frac{\partial\mathrm{v}}{\partial\alpha}+\frac{\partial\mathrm{u}}{\partial\beta}\right)\cos 2\theta=0.$$

The solution of this system of equations for a circular boundary has been considered by Clebsch, (*Theorie der Elasticität fester Körper*, § 74,) the matter is however of little physical interest.

The problem of the equilibrium of a plate under thrusts in its plane will occupy us later in connexion with the stability of the plane form.

CHAPTER XXI.

GENERAL THEORY OF THIN ELASTIC SHELLS.

328. Geometry of the Unstrained Shell.

The kind of body called a "thin shell" may be described as a sort of thin plate which in its natural state is finitely curved. It may be accurately defined in terms of a certain surface to be called the *middle-surface*, and a certain length h, the half-thickness, which is small compared with the unit of length. Thus, taking any surface for middle-surface, and drawing a line from each point of it of very short length $2h$ to be bisected by the surface and to coincide in direction with the normal at the point, we obtain two surfaces near to the middle-surface. If the space between these be supposed to be filled with elastic solid matter in its natural state the body arrived at is a thin shell. In the most general case the half-thickness h may be a function of the position of the point on the middle-surface from which the line of length $2h$ is drawn, and the material may be of any æolotropic and heterogeneous quality. We shall confine our attention to the case of constant thickness and homogeneous isotropic elastic matter.

When the shell is unstrained suppose the lines of curvature on its middle-surface drawn, and let these be the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ Any point in the middle-surface is given by its (α, β), and we shall suppose the length ds of the line joining two points (α, β) and $(\alpha + d\alpha, \beta + d\beta)$ near together to be given by the equation

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2 \quad \text{.....................(1).}$$

Then A and B are some functions of α and β. If a system of axes be constructed at any point (α, β) so that two of the axes touch the curves $\beta = \text{const.}$ and $\alpha = \text{const.}$ and the third is in the normal to the surface, and if the infinitesimal rotations of these axes about themselves as we pass from a point (α, β) to a neighbouring point $(\alpha + d\alpha, \beta + d\beta)$ be

$$p_1 d\alpha + p_2 d\beta, \qquad q_1 d\alpha + q_2 d\beta, \qquad r_1 d\alpha + r_2 d\beta \text{ ......(2)},$$

we shall have the relations of art. 316 in which χ is put equal to $\frac{1}{2}\pi$, and we shall have the further condition that the curves $\beta = \text{const.}$ and $\alpha = \text{const.}$ are lines of curvature.

Now equation (8) of that article shews that in order that the curves in question may be lines of curvature we must have

$$q_2 = 0, \qquad p_1 = 0,$$

and then equation (7) of the same article shews that the principal curvatures $1/\rho_1$ and $1/\rho_2$ are[1]

$$1/\rho_1 = -q_1/A, \qquad 1/\rho_2 = p_2/B.$$

Observing that $p_2 d\beta$ is the angle between the normals to the surface at the extremities of the arc $Bd\beta$ of the line of curvature $\alpha = \text{const.}$, we see that ρ_1 is the radius of curvature in the normal section through $\beta = \text{const.}$, and ρ_2 is the radius of curvature in the normal section through $\alpha = \text{const.}$

The quantities r_1, r_2 are directly given by the fourth and fifth of equations (6) of art. 316, so that we have

$$\left.\begin{aligned} p_1 &= 0, & q_1 &= -\frac{A}{\rho_1}, & r_1 &= -\frac{1}{B}\frac{\partial A}{\partial \beta}, \\ p_2 &= \frac{B}{\rho_2}, & q_2 &= 0, & r_2 &= \frac{1}{A}\frac{\partial B}{\partial \alpha} \end{aligned}\right\} \text{ ....(3).}$$

The sixth of the equations referred to is identically satisfied, and the first three give us

$$\left.\begin{aligned} \frac{\partial}{\partial \alpha}\left(\frac{B}{\rho_2}\right) = \frac{1}{\rho_1}\frac{\partial B}{\partial \alpha}, \qquad \frac{\partial}{\partial \beta}\left(\frac{A}{\rho_1}\right) = \frac{1}{\rho_2}\frac{\partial A}{\partial \beta}, \\ -\frac{AB}{\rho_1 \rho_2} = \frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial B}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial A}{\partial \beta}\right) \end{aligned}\right\} \text{ ....(4).}$$

[1] The radii of curvature are estimated as the z coordinates of the centres of curvature.

329. Kinematics of thin Shells.

When the shell is strained the particles that were originally on the middle-surface come to lie on a new surface which we shall call the *strained middle-surface*, and the particles originally in a line of curvature of the middle-surface come to lie in a line on the strained middle-surface which is not in general one of its lines of curvature. The position of each particle can however be determined by means of its α and β, and in this view α and β are parameters defining the position of a point on the strained middle-surface, but the curves $\alpha = \text{const.}$ and $\beta = \text{const.}$ do not cut at right angles.

Suppose the strained length ds' of the line joining the points (α, β) and $(\alpha + d\alpha, \beta + d\beta)$ to be given by the equation

$$ds'^2 = A'^2 d\alpha^2 + B'^2 d\beta^2 + 2A'B' d\alpha\, d\beta \cos\chi \text{.........(5)},$$

then $A'd\alpha$ is the strained length of the line-element of the curve $\beta = \text{const.}$, $B'd\beta$ is the strained length of the line-element of the curve $\alpha = \text{const.}$, and χ is the angle between the directions of these line-elements after strain.

If we regard only ordinary strains, and take ϵ_1, ϵ_2 for the extensions of these line-elements and ϖ for the shear of their plane, ϵ_1, ϵ_2, and ϖ will be quantities whose squares may be neglected, and we shall have

$$A' = A(1+\epsilon_1), \quad B' = B(1+\epsilon_2), \quad \cos\chi = \varpi, \quad \chi = \tfrac{1}{2}\pi - \varpi \text{...(6)}.$$

Now let a system of rectangular axes of x, y, z be constructed so that when its origin is at any point P of the strained middle-surface the axis x coincides with the line-element $\beta = \text{const.}$ through P, the axis y is in the tangent plane and perpendicular to x, and the axis z is normal to the surface; also let the infinitesimal rotations executed by this system of axes about themselves as we pass from the point (α, β) to the point $(\alpha + d\alpha, \beta + d\beta)$ on the strained middle-surface be

$$p_1' d\alpha + p_2' d\beta, \qquad q_1' d\alpha + q_2' d\beta, \qquad r_1' d\alpha + r_2' d\beta \text{......(7)},$$

then the theory of art. 316 applies to this system.

Now the last three of equations (6) of the article referred to give us

$$\left.\begin{aligned} r_1' &= -\frac{1}{B}\frac{\partial A}{\partial \beta} + \frac{\epsilon_2}{B}\frac{\partial A}{\partial \beta} - \frac{1}{B}\frac{\partial}{\partial \beta}(A\epsilon_1) + \frac{\varpi}{B}\frac{\partial B}{\partial \alpha} + \frac{\partial \varpi}{\partial \alpha}, \\ r_2' &= \frac{1}{A}\frac{\partial B}{\partial \alpha} - \frac{\epsilon_1}{A}\frac{\partial B}{\partial \alpha} + \frac{1}{A}\frac{\partial}{\partial \alpha}(B\epsilon_2) - \frac{\varpi}{A}\frac{\partial A}{\partial \beta}, \\ Aq_2' + Bp_1' &= -A\epsilon_1 q_2' - B\epsilon_2 p_1' + B\varpi q_1' \end{aligned}\right\} \text{...(8)};$$

so that r_1' and r_2' differ from r_1 and r_2, and $q_2'/B+p_1'/A$ differs from zero, by quantities of the order of the extension of the middle-surface.

Now, rejecting small quantities of the order ϵ_1, ϵ_2, ϖ, the equation (7) of art. 316 giving the principal curvatures becomes

$$\frac{1}{\rho^2}-\frac{1}{\rho}\left(\frac{p_2'}{B}-\frac{q_1'}{A}\right)-\left(\frac{p_2'}{B}\frac{q_1'}{A}+\frac{p_1'^2}{A^2}\right)=0,$$

while the third of equations (6) of the same article shews that to the same order of approximation the last term is identical with $-\frac{1}{AB}\left(\frac{\partial r_2}{\partial\alpha}-\frac{\partial r_1}{\partial\beta}\right)$, which gives the same value for the measure of curvature as that before strain. This is merely the well-known result that if a surface be deformed without extension the measure of curvature at any point is unaltered.

If terms of the order ϵ_1, ϵ_2, ϖ be retained, but squares and products of them rejected, the equation giving the principal curvatures becomes[1]

$$\frac{1}{\rho^2}-\frac{1}{\rho}\left(\frac{p_2'}{B}(1-\epsilon_2)-\frac{q_1'}{A}(1-\epsilon_1)-\frac{p_1'\varpi}{A}\right)$$
$$+\left[-\frac{p_2'q_1'}{BA}(1-\epsilon_1-\epsilon_2)-\frac{p_1'^2}{A^2}(1-2\epsilon_1)+\frac{p_1'q_1'\varpi}{A^2}\right]=0\ldots(9).$$

To the same order of approximation the equation giving the directions of the lines of curvature of the strained middle-surface at (α, β) can be shewn to be

$$(x^2-y^2)\frac{p_1'}{A}(1-\epsilon_1)+xy\left\{\frac{p_2'}{B}(1-\epsilon_2)+\frac{q_1'}{A}(1-\epsilon_1)-\frac{\varpi p_1'}{A}\right\}=0\ldots(10).$$

330. Kinematical Equations.

The values of the rotations $p_1'd\alpha, \ldots$ can be determined in particular cases by the use of the following analysis:—

Suppose a system of fixed axes of ξ, η, ζ, and that the system of moving axes of x, y, z at any point (α, β) is connected with this system of fixed axes by the scheme of 9 direction-cosines

$$\left.\begin{array}{cccc} & \xi, & \eta, & \zeta \\ x, & l_1, & m_1, & n_1 \\ y, & l_2, & m_2, & n_2 \\ z, & l_3, & m_3, & n_3 \end{array}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

[1] The student is advised to verify this and the following statement.

Then, just as in art. 318,

$$\left.\begin{aligned}
p_1' &= l_3\frac{\partial l_2}{\partial\alpha} + m_3\frac{\partial m_2}{\partial\alpha} + n_3\frac{\partial n_2}{\partial\alpha},\\
q_1' &= l_1\frac{\partial l_3}{\partial\alpha} + m_1\frac{\partial m_3}{\partial\alpha} + n_1\frac{\partial n_3}{\partial\alpha} = -\left(l_3\frac{\partial l_1}{\partial\alpha} + m_3\frac{\partial m_1}{\partial\alpha} + n_3\frac{\partial n_1}{\partial\alpha}\right),\\
r_1' &= l_2\frac{\partial l_1}{\partial\alpha} + m_2\frac{\partial m_1}{\partial\alpha} + n_2\frac{\partial n_1}{\partial\alpha}
\end{aligned}\right\}\dots(12),$$

and p_2', q_2', r_2' can be obtained from these by writing β for α.

If l_2', m_2', n_2' be the direction-cosines after strain of the line-element which before strain coincided with the line of curvature $\alpha = \text{const.}$, we shall have

$$\left.\begin{aligned}
&l_2' = l_2 + l_1\varpi, && m_2' = m_2 + m_1\varpi, && n_2' = n_2 + n_1\varpi,\\
&Al_1(1+\epsilon_1) = \frac{\partial\xi}{\partial\alpha}, && Am_1(1+\epsilon_1) = \frac{\partial\eta}{\partial\alpha}, && An_1(1+\epsilon_1) = \frac{\partial\zeta}{\partial\alpha},\\
&Bl_2'(1+\epsilon_2) = \frac{\partial\xi}{\partial\beta}, && Bm_2'(1+\epsilon_2) = \frac{\partial\eta}{\partial\beta}, && Bn_2'(1+\epsilon_2) = \frac{\partial\zeta}{\partial\beta}
\end{aligned}\right\}\dots(13),$$

while the direction-cosines l_3, m_3, n_3 of the normal to the strained middle-surface are given by such equations as

$$l_3 = m_1 n_2 - m_2 n_1 \dots\dots\dots\dots(14).$$

The above equations give us

$$\left.\begin{aligned}
(1+\epsilon_1)^2 &= \frac{1}{A^2}\left[\left(\frac{\partial\xi}{\partial\alpha}\right)^2 + \left(\frac{\partial\eta}{\partial\alpha}\right)^2 + \left(\frac{\partial\zeta}{\partial\alpha}\right)^2\right],\\
(1+\epsilon_2)^2 &= \frac{1}{B^2}\left[\left(\frac{\partial\xi}{\partial\beta}\right)^2 + \left(\frac{\partial\eta}{\partial\beta}\right)^2 + \left(\frac{\partial\zeta}{\partial\beta}\right)^2\right],\\
(1+\epsilon_1)(1+\epsilon_2)\varpi &= \frac{1}{AB}\left[\frac{\partial\xi}{\partial\alpha}\frac{\partial\xi}{\partial\beta} + \frac{\partial\eta}{\partial\alpha}\frac{\partial\eta}{\partial\beta} + \frac{\partial\zeta}{\partial\alpha}\frac{\partial\zeta}{\partial\beta}\right]
\end{aligned}\right\}\dots(15).$$

331. Small Displacements. The Extension.

We shall now suppose that the thin shell is very slightly deformed so that any particle P initially on the middle-surface undergoes displacements u, v, w in the directions of the lines of curvature and the normal through its equilibrium position, u being parallel to the tangent to the line of curvature $\beta = \text{const.}$, and v to the line of curvature $\alpha = \text{const.}$ The lines of curvature and the normal through any point are the lines of reference for u, v, w.

Through the point occupied by any particle P of the middle-surface before strain draw a system of fixed axes of ξ, η, ζ coinciding with the lines of reference for u, v, w at P, (α, β). The lines of reference for u, v, w at a neighbouring point P' $(\alpha + d\alpha, \beta + d\beta)$ are to be obtained by translations of the origin through $Ad\alpha$, $Bd\beta$ parallel to the axes of ξ, η, and a rotation whose components about the axes of ξ, η, ζ are

$$\frac{Bd\beta}{\rho_2}, \quad -\frac{Ad\alpha}{\rho_1}, \quad -\frac{\partial A}{\partial \beta}\frac{d\alpha}{B} + \frac{\partial B}{\partial \alpha}\frac{d\beta}{A} \text{......(16).}$$

Let ξ, η, ζ be the coordinates of P after strain, and $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$ those of P'; then ξ, η, ζ are identical with u, v, w, but $d\xi$, $d\eta$, $d\zeta$ are not identical with du, dv, dw, since u, v, w are referred to moving axes. In fact we have such formulæ as

$$d\xi = Ad\alpha + d\mathrm{u} - \mathrm{v}\left(-\frac{\partial A}{\partial \beta}\frac{d\alpha}{B} + \frac{\partial B}{\partial \alpha}\frac{d\beta}{A}\right) + \mathrm{w}\left(-\frac{Ad\alpha}{\rho_1}\right),$$

in which any differential as du means $(\partial \mathrm{u}/\partial \alpha)\, d\alpha + (\partial \mathrm{u}/\partial \beta)\, d\beta$.

Writing these formulæ down, and equating coefficients of $d\alpha$, $d\beta$, we find

$$\left.\begin{aligned}
\frac{\partial \xi}{\partial \alpha} &= A + \frac{\partial \mathrm{u}}{\partial \alpha} + \frac{\mathrm{v}}{B}\frac{\partial A}{\partial \beta} - A\frac{\mathrm{w}}{\rho_1}, & \frac{\partial \xi}{\partial \beta} &= \frac{\partial \mathrm{u}}{\partial \beta} - \frac{\mathrm{v}}{A}\frac{\partial B}{\partial \alpha}, \\
\frac{\partial \eta}{\partial \alpha} &= \frac{\partial \mathrm{v}}{\partial \alpha} - \frac{\mathrm{u}}{B}\frac{\partial A}{\partial \beta}, & \frac{\partial \eta}{\partial \beta} &= B + \frac{\partial \mathrm{v}}{\partial \beta} + \frac{\mathrm{u}}{A}\frac{\partial B}{\partial \alpha} - B\frac{\mathrm{w}}{\rho_2}, \\
\frac{\partial \zeta}{\partial \alpha} &= \frac{\partial \mathrm{w}}{\partial \alpha} + A\frac{\mathrm{u}}{\rho_1}, & \frac{\partial \zeta}{\partial \beta} &= \frac{\partial \mathrm{w}}{\partial \beta} + B\frac{\mathrm{v}}{\rho_2}
\end{aligned}\right\} \text{(17).}$$

From equations (15) of art. 330 we now find, on rejecting squares of small quantities, the following values for the quantities ϵ_1, ϵ_2, ϖ defining the extension of the middle-surface:

$$\left.\begin{aligned}
\epsilon_1 &= \frac{1}{A}\frac{\partial \mathrm{u}}{\partial \alpha} + \frac{\mathrm{v}}{AB}\frac{\partial A}{\partial \beta} - \frac{\mathrm{w}}{\rho_1}, \\
\epsilon_2 &= \frac{1}{B}\frac{\partial \mathrm{v}}{\partial \beta} + \frac{\mathrm{u}}{AB}\frac{\partial B}{\partial \alpha} - \frac{\mathrm{w}}{\rho_2}, \\
\varpi &= \frac{1}{A}\frac{\partial \mathrm{v}}{\partial \alpha} + \frac{1}{B}\frac{\partial \mathrm{u}}{\partial \beta} - \frac{\mathrm{u}}{AB}\frac{\partial A}{\partial \beta} - \frac{\mathrm{v}}{AB}\frac{\partial B}{\partial \alpha}
\end{aligned}\right\} \text{......(18)[1].}$$

[1] Expressions equivalent to these were given in my paper (*Phil. Trans.* A. 1888). Another method of arriving at them has been given by Prof. Lamb in *Proc. Lond. Math. Soc.* XXI. 1890. The expressions found below for the quantities that define the changes of curvature of the middle-surface have a similar history.

332. Formulæ for the changes of curvature.

We have next to calculate the quantities $p_1', q_1', r_1', p_2', q_2', r_2'$ defined by equations (12) of art. 330. We shall suppose that the moving axes of x, y, z introduced in art. 329, when referred to the lines of reference for u, v, w, are given by the scheme of 9 direction-cosines

$$\left.\begin{array}{cccc} & \mathrm{u}, & \mathrm{v}, & \mathrm{w} \\ x, & \lambda_1, & \mu_1, & \nu_1 \\ y, & \lambda_2, & \mu_2, & \nu_2 \\ z, & \lambda_3, & \mu_3, & \nu_3 \end{array}\right\} \text{......................} (19),$$

while, referred to the axes of ξ, η, ζ they are given by the scheme

$$\left.\begin{array}{cccc} & \xi, & \eta, & \zeta \\ x, & l_1, & m_1, & n_1 \\ y, & l_2, & m_2, & n_2 \\ z, & l_3, & m_3, & n_3 \end{array}\right\} \text{......................} (20).$$

Then $\lambda_1 \dots$ coincide with $l_1 \dots$, but $d\lambda_1 \dots$ differ from $dl_1 \dots$ because $\lambda_1 \dots$ are referred to moving axes.

From equations (13) of art. 330 we find $l_1 \dots$ in terms of the coordinates ξ, η, ζ, *i.e.* we find $\lambda_1 \dots$ in terms of the displacements u, v, w. Using (18) we obtain the results

$$\left.\begin{array}{lll} \lambda_1 = 1, & \mu_1 = \dfrac{1}{A}\dfrac{\partial \mathrm{v}}{\partial \alpha} - \dfrac{\mathrm{u}}{AB}\dfrac{\partial A}{\partial \beta}, & \nu_1 = \dfrac{1}{A}\dfrac{\partial \mathrm{w}}{\partial \alpha} + \dfrac{\mathrm{u}}{\rho_1}, \\ \lambda_2 = -\dfrac{1}{A}\dfrac{\partial \mathrm{v}}{\partial \alpha} + \dfrac{\mathrm{u}}{AB}\dfrac{\partial A}{\partial \beta}, & \mu_2 = 1, & \nu_2 = \dfrac{1}{B}\dfrac{\partial \mathrm{w}}{\partial \beta} + \dfrac{\mathrm{v}}{\rho_2}, \\ \lambda_3 = -\dfrac{1}{A}\dfrac{\partial \mathrm{w}}{\partial \alpha} - \dfrac{\mathrm{u}}{\rho_1}, & \mu_3 = -\dfrac{1}{B}\dfrac{\partial \mathrm{w}}{\partial \beta} - \dfrac{\mathrm{v}}{\rho_2}, & \nu_3 = 1 \end{array}\right\} \dots (21).$$

Now, since at (α, β) $\lambda_1, \dots$ are identical with $l_1, \dots$, and since the axes of reference for $\lambda_1, \dots$ at a neighbouring point are obtained from the axes at α, β by the rotations (16) we have such formulæ as

$$dl_1 = d\lambda_1 - \mu_1\left(-\frac{\partial A}{\partial \beta}\frac{d\alpha}{B} + \frac{\partial B}{\partial \alpha}\frac{d\beta}{A}\right) + \nu_1\left(-\frac{A d\alpha}{\rho_1}\right).$$

We can hence write down the partial differential coefficients of $l_1, \dots$ with respect to α, β. But as we only require certain expressions involving them, and, as in these expressions we shall reject all terms which contain products of u, v, w or their differential coefficients, it is shorter to pick out the terms that arise

without writing down all the formulæ. We can in this way calculate the quantities p_1', ... from equations (12) of art. 330.

We have for example

$$p_1' = l_3 \frac{\partial l_2}{\partial \alpha} + m_3 \frac{\partial m_2}{\partial \alpha} + n_3 \frac{\partial n_2}{\partial \alpha}.$$

In this formula l_3 is of the first order in u, v, w; we need therefore only retain the terms, if any, of $\partial l_2/\partial \alpha$ that are independent of u, v, w; also we have

$$dl_2 = d\lambda_2 - \mu_2 \left(-\frac{\partial A}{\partial \beta} \frac{d\alpha}{B} + \frac{\partial B}{\partial \alpha} \frac{d\beta}{A} \right) + \nu_2 \left(-\frac{A d\alpha}{\rho_1} \right),$$

in which $d\lambda_2$ and ν_2 are linear in u, v, w, and the only term which need be retained is the term in μ_2. Proceeding in this way we find

$$\left.\begin{aligned} p_1' &= -\frac{1}{B}\frac{\partial A}{\partial \beta}\left(\frac{1}{A}\frac{\partial \mathrm{w}}{\partial \alpha} + \frac{\mathrm{u}}{\rho_1}\right) + \frac{\partial}{\partial \alpha}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial \beta} + \frac{\mathrm{v}}{\rho_2}\right) \\ &\qquad + \frac{A}{\rho_1}\left(-\frac{1}{A}\frac{\partial \mathrm{v}}{\partial \alpha} + \frac{\mathrm{u}}{AB}\frac{\partial A}{\partial \beta}\right), \\ q_1' &= -\frac{1}{B}\frac{\partial A}{\partial \beta}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial \beta} + \frac{\mathrm{v}}{\rho_2}\right) - \frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial \mathrm{w}}{\partial \alpha} + \frac{\mathrm{u}}{\rho_1}\right) - \frac{A}{\rho_1}, \\ r_1' &= \frac{\partial}{\partial \alpha}\left(\frac{1}{A}\frac{\partial \mathrm{v}}{\partial \alpha} - \frac{\mathrm{u}}{AB}\frac{\partial A}{\partial \beta}\right) + \frac{A}{\rho_1}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial \beta} + \frac{\mathrm{v}}{\rho_2}\right) - \frac{1}{B}\frac{\partial A}{\partial \beta} \end{aligned}\right\} \ldots (22),$$

and

$$\left.\begin{aligned} p_2' &= \frac{1}{A}\frac{\partial B}{\partial \alpha}\left(\frac{1}{A}\frac{\partial \mathrm{w}}{\partial \alpha} + \frac{\mathrm{u}}{\rho_1}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial \beta} + \frac{\mathrm{v}}{\rho_2}\right) + \frac{B}{\rho_2}, \\ q_2' &= \frac{1}{A}\frac{\partial B}{\partial \alpha}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial \beta} + \frac{\mathrm{v}}{\rho_2}\right) - \frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial \mathrm{w}}{\partial \alpha} + \frac{\mathrm{u}}{\rho_1}\right) \\ &\qquad - \frac{B}{\rho_2}\left(\frac{1}{A}\frac{\partial \mathrm{v}}{\partial \alpha} - \frac{\mathrm{u}}{AB}\frac{\partial A}{\partial \beta}\right), \\ r_2' &= \frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial \mathrm{v}}{\partial \alpha} - \frac{\mathrm{u}}{AB}\frac{\partial A}{\partial \beta}\right) - \frac{B}{\rho_2}\left(\frac{1}{A}\frac{\partial \mathrm{w}}{\partial \alpha} + \frac{\mathrm{u}}{\rho_1}\right) + \frac{1}{A}\frac{\partial B}{\partial \alpha} \end{aligned}\right\} \ldots (23).$$

The results here obtained admit of certain verifications[1] as follows:

1°. With the values for ϵ_1, ϵ_2, ϖ given in equations (18) of art. 331 the equations (8) of art. 329 are identically satisfied, squares of u, v, w being neglected.

[1] The student is recommended to work these out.

2°. The surface $r = a + bP_n$, in which P_n is Legendre's nth coefficient and b is small, being supposed to be derived from a sphere of radius a by the normal displacement bP_n, the expressions for the sum of the curvatures and the measure of curvature given by equation (9) of art. 329 will be found to coincide with the known expressions, viz. we have

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = -\frac{2}{a} - \frac{b}{a^2}(n-1)(n+2)P_n,$$

$$\frac{1}{\rho_1\rho_2} = \frac{1}{a^2} + \frac{b}{a^3}(n-1)(n+2)P_n,$$

the argument of P_n being the sine of the latitude of a point on the undeformed sphere.

It will hereafter be found that for determining the strain in an element of the shell the quantities of greatest importance are ϵ_1, ϵ_2, ϖ which define the stretching of the middle-surface, and three quantities κ_1, κ_2, τ defined by the equations

$$\kappa_1 = -\frac{q_1'}{A} - \frac{1}{\rho_1}, \qquad \kappa_2 = \frac{p_2'}{B} - \frac{1}{\rho_2}, \qquad \tau = \frac{p_1'}{A} \quad \text{......(24).}$$

These quantities κ_1, κ_2, τ are sufficient in the case of bending of the middle-surface accompanied by very slight stretching to define the bending. We shall refer to them as the *changes of curvature.*

According to equations (22) and (23) we have for κ_1, κ_2, τ in terms of the displacements of a point on the middle-surface

$$\left.\begin{aligned}
\kappa_1 &= \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial \mathrm{w}}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial \mathrm{w}}{\partial\beta} + \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{\mathrm{u}}{\rho_1}\right) + \frac{\mathrm{v}}{AB\rho_2}\frac{\partial A}{\partial\beta},\\
\kappa_2 &= \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial\beta}\right) + \frac{1}{BA^2}\frac{\partial B}{\partial\alpha}\frac{\partial \mathrm{w}}{\partial\alpha} + \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{\mathrm{v}}{\rho_2}\right) + \frac{\mathrm{u}}{AB\rho_1}\frac{\partial B}{\partial\alpha},\\
\tau &= \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\frac{\partial \mathrm{w}}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial \mathrm{w}}{\partial\alpha} + \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{\mathrm{v}}{\rho_2}\right) - \frac{1}{A\rho_1}\frac{\partial \mathrm{v}}{\partial\alpha}
\end{aligned}\right\}\text{...(25).}$$

333. Extension of Kirchhoff's Theory to Thin Shells.

We have in the next place to give a theory for determining to a sufficient order of approximation the stress-resultants and stress-couples that act upon any element of the shell. We cannot as in the corresponding case of naturally curved wires obtain these by a process of superposition. When a curved wire is bent and twisted into a new curved form we may suppose the operation conducted

by first straightening and untwisting the wire into a straight prismatic rod, and then bending and twisting it into the new shape; but it is different with a thin shell. Unless its middle-surface be a developable it cannot be unbent into a plane plate without producing more than ordinary strains, and we may not therefore suppose that the thin shell is first strained into a plane plate and then strained again into its new form.

We shall adapt the process of arts. 319 sq. to obtain the theory of which we are in search, and the first step in this process is the investigation of certain differential identities expressing the conditions that after strain the elements of the shell continue to form a continuous shell.

We take P to be a point on the middle-surface, and suppose that three line-elements (1, 2, 3) of the shell proceed from P, of which (1) and (2) lie along the lines of curvature of the unstrained middle-surface, $\beta = \text{const.}$ and $\alpha = \text{const.}$, while (3) lies along the normal. After strain these are not in general rectangular, but by means of them we construct, as in art. 329, a system of rectangular axes of x, y, z to which we can refer points in the neighbourhood of P; viz. the new position of P is to be the origin, the new position of the line-element (1) the axis x, and the new plane of the line-elements (1) and (2) the plane of (x, y).

334. Differential Identities.

We have to consider the strain in an element of the shell bounded by normal sections through two pairs of lines of curvature whose distances are comparable with the thickness of the shell. We shall suppose P to be the centre of such an element, and we may refer to the element as the prism whose centre is P. After the deformation we can suppose this prism moved back without strain so that the axes of x, y, z come to lie along the initial positions of the line-elements (1, 2, 3).

Now let Q be a point of the shell near to P, and before the deformation let x, y, z be the coordinates of Q referred to the line-elements at P. After strain let $x+u$, $y+v$, $z+w$ be the coordinates of Q referred to the axes x, y, z; then u, v, w are the displacements of the point Q relative to P, and they are functions of x, y, z, α, β.

Suppose P' is a point on the middle-surface near to P, then we might refer Q to the line-elements at P', coinciding with the lines of curvature and the normal at P', instead of to the line-elements at P. The coordinates of Q before strain referred to P' are $x+\delta x$, $y+\delta y$, $z+\delta z$, where

$$\left.\begin{aligned}\delta x &= -A\,d\alpha + y\,(r_1 d\alpha + r_2 d\beta) - z\,(q_1 d\alpha + q_2 d\beta)\\ &\qquad = -A\,d\alpha + y\,(r_1 d\alpha + r_2 d\beta) + Az\,d\alpha/\rho_1,\\ \delta y &= -B\,d\beta + z\,(p_1 d\alpha + p_2 d\beta) - x\,(r_1 d\alpha + r_2 d\beta)\\ &\qquad = -B\,d\beta + Bz\,d\beta/\rho_2 - x\,(r_1 d\alpha + r_2 d\beta),\\ \delta z &= \qquad x\,(q_1 d\alpha + q_2 d\beta) - y\,(p_1 d\alpha + p_2 d\beta)\\ &\qquad = -Ax\,d\alpha/\rho_1 - By\,d\beta/\rho_2\end{aligned}\right\}\quad\ldots(26),$$

since the line-elements at P' are obtained from those at P by infinitesimal translations $A\,d\alpha$, $B\,d\beta$ and infinitesimal rotations $p_1 d\alpha + p_2 d\beta$, ..., in which the p's, ... are given by (3). To obtain the values of the coordinates of Q referred to P' we must in the expressions for u, v, w replace α and β by $\alpha + d\alpha$ and $\beta + d\beta$, and replace x, y, z by $x+\delta x$, $y+\delta y$, $z+\delta z$, where δx, δy, δz are given by the equations (26). Thus we find for the coordinates of Q referred to P' such expressions as

$$x + \delta x + u + \frac{\partial u}{\partial \alpha}\,d\alpha + \frac{\partial u}{\partial \beta}\,d\beta + \frac{\partial u}{\partial x}\,\delta x + \frac{\partial u}{\partial y}\,\delta y + \frac{\partial u}{\partial z}\,\delta z \ldots(27),$$

in which δx, δy, δz are linear functions of $d\alpha$, $d\beta$ given by (26).

Again the axes of x, y, z at P' after strain are obtained from those at P by translations

$$A\,(1+\epsilon_1)\,d\alpha + B\varpi\,d\beta, \qquad B\,(1+\epsilon_2)\,d\beta$$

parallel to the axes of x, y at P, and by rotations

$$p_1' d\alpha + p_2' d\beta, \qquad q_1' d\alpha + q_2' d\beta, \qquad r_1' d\alpha + r_2' d\beta$$

about the axes of x, y, z. Hence the coordinates of Q referred to P' after strain are

$$\left.\begin{aligned}&x + u + (y+v)\,(r_1' d\alpha + r_2' d\beta) - (z+w)\,(q_1' d\alpha + q_2' d\beta)\\ &\qquad\qquad - A\,(1+\epsilon_1)\,d\alpha - B\varpi\,d\beta,\\ &y + v + (z+w)(p_1' d\alpha + p_2' d\beta) - (x+u)\,(r_1' d\alpha + r_2' d\beta)\\ &\qquad\qquad - B\,(1+\epsilon_2)\,d\beta,\\ &z + w + (x+u)\,(q_1' d\alpha + q_2' d\beta) - (y+v)\,(p_1' d\alpha + p_2' d\beta)\end{aligned}\right\}\ldots(28).$$

Comparing the expressions (27) and (28) for the coordinates of Q referred to P', and using the values of δx, δy, δz given in (26), we see that we shall obtain three equations linear in $d\alpha$, $d\beta$, which must be satisfied for all values of the ratio $d\alpha : d\beta$. We may therefore equate coefficients of $d\alpha$ and $d\beta$, and we thus obtain six equations, viz.:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial x}\{A(1-z/\rho_1)-r_1y\}+\frac{\partial u}{\partial y}r_1x+\frac{\partial u}{\partial z}Ax/\rho_1\\
&\qquad=\frac{\partial u}{\partial \alpha}-\{(r_1'-r_1)y+r_1'v\}+\{(q_1'+A/\rho_1)z+q_1'w\}+A\epsilon_1,\\
&\frac{\partial v}{\partial x}\{A(1-z/\rho_1)-r_1y\}+\frac{\partial v}{\partial y}r_1x+\frac{\partial v}{\partial z}Ax/\rho_1\\
&\qquad=\frac{\partial v}{\partial \alpha}-p_1'(z+w)+\{(r_1'-r_1)x+r_1'u\},\\
&\frac{\partial w}{\partial x}\{A(1-z/\rho_1)-r_1y\}+\frac{\partial w}{\partial y}r_1x+\frac{\partial w}{\partial z}Ax/\rho_1\\
&\qquad=\frac{\partial w}{\partial \alpha}-\{(q_1'+A/\rho_1)x+q_1'u\}+p_1'(y+v)
\end{aligned}\right\}\ldots(29),$$

and

$$\left.\begin{aligned}
&-\frac{\partial u}{\partial x}r_2y+\frac{\partial u}{\partial y}\{B(1-z/\rho_2)+r_2x\}+\frac{\partial u}{\partial z}By/\rho_2\\
&\qquad=\frac{\partial u}{\partial \beta}-\{(r_2'-r_2)y+r_2'v\}+q_2'(z+w)+B\varpi,\\
&-\frac{\partial v}{\partial x}r_2y+\frac{\partial v}{\partial y}\{B(1-z/\rho_2)+r_2x\}+\frac{\partial v}{\partial z}By/\rho_2\\
&\qquad=\frac{\partial v}{\partial \beta}-\{(p_2'-B/\rho_2)z+p_2'w\}+\{(r_2'-r_2)x+r_2'u\}+B\epsilon_2,\\
&-\frac{\partial w}{\partial x}r_2y+\frac{\partial w}{\partial y}\{B(1-z/\rho_2)+r_2x\}+\frac{\partial w}{\partial z}By/\rho_2\\
&\qquad=\frac{\partial w}{\partial \beta}-q_2'(x+u)+\{(p_2'-B/\rho_2)y+p_2'v\}
\end{aligned}\right\}\ldots(30).$$

For a plane plate these reduce to (20) and (21) of art. 320.

335. First Approximation.

We have precisely the same classification of the modes of deformation of thin shells as of those of thin plates, and we may treat

the equations just obtained in precisely the same way as the corresponding equations of art. 320 were treated. For the first approximation we leave out terms like $A\frac{z}{\rho_1}\frac{\partial u}{\partial x}$ as being of the same order as u, we leave out terms like $\frac{\partial u}{\partial \alpha}$, and we leave out terms like $q_1'w$; also we leave out terms like $(r_1'-r_1)y$, because $(r_1'-r_1)$ is of the same order as the extension of the middle-surface, and we replace q_2'/B by $-p_1'/A$ because the difference of these two quantities is small of the same order.

The approximate equations which replace (29) and (30) are therefore

$$\left.\begin{array}{ll}\dfrac{\partial u}{\partial x}=\epsilon_1+\left(\dfrac{q_1'}{A}+\dfrac{1}{\rho_1}\right)z, & \dfrac{\partial u}{\partial y}=\varpi-\dfrac{p_1'}{A}z,\\ \dfrac{\partial v}{\partial x}=\quad-\dfrac{p_1'}{A}z, & \dfrac{\partial v}{\partial y}=\epsilon_2-\left(\dfrac{p_2'}{B}-\dfrac{1}{\rho_2}\right)z,\\ \dfrac{\partial w}{\partial x}=\quad-\left(\dfrac{q_1'}{A}+\dfrac{1}{\rho_1}\right)x+\dfrac{p_1'}{A}y, & \dfrac{\partial w}{\partial y}=\quad\dfrac{p_1'}{A}x+\left(\dfrac{p_2'}{B}-\dfrac{1}{\rho_2}\right)y\end{array}\right\}\ldots(31).$$

If now we write, as in (25) of art. 332,

$$\kappa_1=-\left(\frac{q_1'}{A}+\frac{1}{\rho_1}\right),\qquad \kappa_2=\left(\frac{p_2'}{B}-\frac{1}{\rho_2}\right),\qquad \tau=\frac{p_1'}{A}$$

we have precisely the same system of equations for the displacements within an element as in the corresponding case of a plane plate, and they lead to precisely the same conclusions as regards the strain and stress within an element of the shell.

As far as terms in z we find accordingly that the strains are

$$\left.\begin{array}{ll}e=\epsilon_1-\kappa_1 z, & a=0,\\ f=\epsilon_2-\kappa_2 z, & b=0,\\ g=-\dfrac{\lambda}{\lambda+2\mu}\{\epsilon_1+\epsilon_2-(\kappa_1+\kappa_2)z\}, & c=\varpi-2\tau z\end{array}\right\}\ldots(32).$$

As far as quadratic terms in x, y, z the displacements are

$$\left.\begin{array}{l}u=-\kappa_1 zx-\tau zy+\epsilon_1 x+\varpi y,\\ v=-\tau zx-\kappa_2 zy+\epsilon_2 y,\\ w=-\dfrac{\lambda}{\lambda+2\mu}\left[(\epsilon_1+\epsilon_2)z-\tfrac{1}{2}(\kappa_1+\kappa_2)z^2\right]+\tfrac{1}{2}(\kappa_1 x^2+\kappa_2 y^2+2\tau xy)\end{array}\right\}\ldots(33).$$

In obtaining these the equations of equilibrium or small motion are simplified by omitting bodily forces and kinetic reactions for the reason explained in art. 249.

The above is a sufficient approximation for the purpose of finding the values of the stress-couples, but it is generally insufficient for the purpose of finding the values of the stress-resultants.

When a second approximation is made the complete forms of the differential equations of equilibrium or small motion including bodily forces and kinetic reactions will have to be used for the determination of the arbitrary functions or constants introduced by integration. Now the second approximations to e, f, c are found directly by substituting the approximations already found in the terms of equations (29) and (30) previously rejected, so that the values of e, f, c do not depend on the equations of equilibrium. This observation shews that a second approximation to the strains e, f, c and the stress U can be found without having recourse to these equations, but the other stresses cannot be found in the same way.

336. Second Approximation.

It is necessary to go through the second approximation so far as to find expressions for the strains e, f, c which can be obtained without having recourse to the equations of equilibrium. Certain simplifications are however possible. The resultant stresses which we have to find may be calculated for an indefinitely narrow strip of the normal section along the line $x=0$, $y=0$, and we may therefore confine our attention to the values of the strains when x and y vanish but z is variable. The expressions we have already found give the first two terms of an expansion of these strains in powers of z, and we propose to find the terms in z^2 in the same expansions. Now from the classification of cases in art. 321, it appears that different cases are discriminated by the varying relative importance of such quantities as ϵ_1 and such quantities as $\kappa_1 z$, $\kappa_1 z^2$, ... When ϵ_1 is great compared with $\kappa_1 z$, only the first terms of the expansion (those independent of z) need for any purpose be retained. When ϵ_1 is of the same order as $\kappa_1 z$, only the first two terms need be retained. The second approximation

therefore refers only to cases in which ϵ_1 is small compared with $\kappa_1 z$. In making the second approximation we are at liberty to reject all such terms as $\epsilon_1 z$, $\epsilon_2 z$, ϖz, $\epsilon_1 z^2$, ... as well as all terms that contain x or y. Further, since our second approximation will only be applied to small displacements of the middle-surface we may reject all terms which involve products of κ_1, κ_2, τ.

Looking at the system of equations (29) and (30) we see that of the terms at first rejected $\frac{\partial u}{\partial \alpha}$, $\frac{\partial v}{\partial \alpha}$, $\frac{\partial u}{\partial \beta}$, $\frac{\partial v}{\partial \beta}$ may still be omitted, for our purpose, because every term of each of them contains a factor x or y. Picking out at first all the terms that do not vanish with x and y we have such equations as

$$\frac{\partial u}{\partial x} = \frac{\epsilon_1}{1 - z/\rho_1} + \frac{1}{1 - z/\rho_1}\left[\left(\frac{q_1'}{A} + \frac{1}{\rho_1}\right) z - \frac{q_1'}{A}\frac{\sigma}{1-\sigma}\{(\epsilon_1 + \epsilon_2) z - \tfrac{1}{2}(\kappa_1 + \kappa_2) z^2\}\right],$$

and, in accordance with the principles above explained, these may be simplified to

$$\left.\begin{aligned} e &= \frac{\partial u}{\partial x} = \epsilon_1 - \kappa_1 z - \frac{\kappa_1 z^2}{\rho_1} - \tfrac{1}{2}\frac{z^2}{\rho_1}\frac{\sigma}{1-\sigma}(\kappa_1 + \kappa_2), \\ f &= \frac{\partial v}{\partial y} = \epsilon_2 - \kappa_2 z - \frac{\kappa_2 z^2}{\rho_2} - \tfrac{1}{2}\frac{z^2}{\rho_2}\frac{\sigma}{1-\sigma}(\kappa_1 + \kappa_2) \end{aligned}\right\} \text{......(34).}$$

Also in the same way we find

$$\begin{aligned} \frac{\partial v}{\partial x} &= \quad - \tau z - \frac{\tau z^2}{\rho_1}, \\ \frac{\partial u}{\partial y} &= \varpi - \tau z - \frac{\tau z^2}{\rho_2}, \end{aligned}$$

so that

$$c = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \varpi - 2\tau z - \tau z^2\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \text{ ....(35).}$$

The above equations give the strains e, f, c correct to terms in z^2, when $x = 0$, $y = 0$ and the extension of the middle-surface is small compared with the products of the changes of curvature and the thickness, while the displacement of any point of the middle-surface is small enough for its square to be neglected. In obtaining them we have used equations (24) and written $\sigma/(1 - \sigma)$ for $\lambda/(\lambda + 2\mu)$.

337. Stress-components.

We now consider the stress-system. As far as terms in z we have by the method of art. 323

$$\left.\begin{aligned} P &= \frac{3C}{2h^3}\{\epsilon_1+\sigma\epsilon_2 - z(\kappa_1+\sigma\kappa_2)\}, & R&=0,\\ Q &= \frac{3C}{2h^3}\{\epsilon_2+\sigma\epsilon_1 - z(\kappa_2+\sigma\kappa_1)\}, & S&=0,\\ U &= \frac{3C(1-\sigma)}{4h^3}\{\varpi-2\tau z\}, & T&=0 \end{aligned}\right\} \quad \text{......(36)},$$

where C is the cylindrical rigidity $\dfrac{8}{3}\mu h^3\dfrac{\lambda+\mu}{\lambda+2\mu}$.

We cannot obtain the terms in z^2, except in U, without having recourse to the equations of equilibrium, but we can give a form for the results as far as P and Q are concerned.

Suppose the term in R that contains z^2 is $R_0 z^2$, then

$$\lambda(e+f+g)+2\mu g = R_0 z^2$$

so that
$$g = -\frac{\lambda}{\lambda+2\mu}(e+f)+\frac{R_0 z^2}{\lambda+2\mu}.$$

Hence

$$P = (\lambda+2\mu)e+\lambda f - \frac{\lambda^2}{\lambda+2\mu}(e+f)+\frac{\lambda}{\lambda+2\mu}R_0 z^2,$$

or
$$P = \frac{3C}{2h^3}\left\{e+\sigma f+\tfrac{1}{2}\frac{\sigma R_0}{\mu}z^2\right\}.$$

So
$$Q = \frac{3C}{2h^3}\left\{f+\sigma e+\tfrac{1}{2}\frac{\sigma R_0}{\mu}z^2\right\}.$$

Thus we find from equations (34)

$$\left.\begin{aligned} P &= \frac{3C}{2h^3}\Big[(\epsilon_1+\sigma\epsilon_2)-(\kappa_1+\sigma\kappa_2)z-\left(\frac{\kappa_1}{\rho_1}+\frac{\sigma\kappa_2}{\rho_2}\right)z^2\\ &\qquad -\tfrac{1}{2}\frac{\sigma}{1-\sigma}(\kappa_1+\kappa_2)\left(\frac{1}{\rho_1}+\frac{\sigma}{\rho_2}\right)z^2+\tfrac{1}{2}\frac{\sigma R_0}{\mu}z^2\Big],\\ Q &= \frac{3C}{2h^3}\Big[(\epsilon_2+\sigma\epsilon_1)-(\kappa_2+\sigma\kappa_1)z-\left(\frac{\kappa_2}{\rho_2}+\frac{\sigma\kappa_1}{\rho_1}\right)z^2\\ &\qquad -\tfrac{1}{2}\frac{\sigma}{1-\sigma}(\kappa_1+\kappa_2)\left(\frac{\sigma}{\rho_1}+\frac{1}{\rho_2}\right)z^2+\tfrac{1}{2}\frac{\sigma R_0}{\mu}z^2\Big],\\ &\text{and from equation (35)}\\ U &= \frac{3C(1-\sigma)}{4h^3}\left[\varpi-2\tau z-\tau z^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\right] \end{aligned}\right\} \text{...(37)}.$$

338. Potential Energy of strained Shell.

From the approximate values of the strains in art. 335 we may deduce an expression for the potential energy. Just as in art. 323 it appears that the potential energy per unit area of the strained shell is

$$\tfrac{1}{2}C\left[(\kappa_1+\kappa_2)^2-2(1-\sigma)(\kappa_1\kappa_2-\tau^2)\right]$$
$$+\frac{3C}{2h^2}\left[(\epsilon_1+\epsilon_2)^2-2(1-\sigma)(\epsilon_1\epsilon_2-\tfrac{1}{4}\varpi^2)\right] \quad\text{.........}\ (38).$$

The second term of this expression contains h as a factor, and the first term contains h^3. If more exact values of the strains were found there would be additional terms in h^3 depending on products of such quantities as ϵ_1 and κ_1. When the middle-surface undergoes extension as well as flexure the second line is the most important. When the extension is small in such a way that the quantities ϵ_1, ϵ_2, ϖ are of the same order of magnitude as $\kappa_1 h$, $\kappa_2 h$, τh, the two lines must be retained. When ϵ_1, ϵ_2, ϖ are of the order $\kappa_1 h^2$, $\kappa_2 h^2$, τh^2 or of any higher order the first line is the most important. In each class of cases the expression above written includes all the terms that need to be retained (and generally some terms that ought to be rejected) for the purpose of estimating the potential energy. It is necessary to observe that the equations of equilibrium or small motion cannot be deduced from the above expression by the method of variation[1], because the terms in h^3 that are neglected therein may on variation give rise to terms of the same order of magnitude as those that arise from the terms retained. For instance, from the variation of a term in $h^3\kappa_1\epsilon_1$ there would proceed a term $h^3\kappa_1\delta\epsilon_1$ as well as a term $h^3\epsilon_1\delta\kappa_1$, and the former would give us terms in $h^3\kappa_1$ in the equations of equilibrium or small motion which are of the same order as those that come from the variation of $h^3\kappa_1^2$. We shall obtain the general equations without using the method of variation.

[1] This was the method employed in my paper, (*Phil. Trans.* A. 1888,) its inexactness was pointed out by Mr Basset (*Phil. Trans.* A. 1890). If the general equations of my paper be compared with those of any of the problems solved below it will be found that terms are omitted which ought to have been retained. These terms depend on the second approximations to the stresses considered in the last article. Lord Rayleigh has also pointed out that when extension of the middle-surface takes place there does not exist a function depending on the deformation of the middle-surface alone which represents the potential energy correct to terms in h^3. ('On the Uniform Deformation...of a Cylindrical Shell...' *Proc. Lond. Math. Soc.* xx. 1889.)

339. Stress-resultants and Stress-couples.

For the purpose of forming the equations of equilibrium of an element of the shell bounded by normal sections through two pairs of consecutive lines of curvature we seek to reduce the stresses on the faces of the element to force- and couple-resultants. Each of these will be estimated per unit of length of the trace on the middle-surface of the face across which it acts.

On the element of the face $\alpha = \text{const.}$ whose trace on the middle-surface is of length $Bd\beta$ there act forces of the P, U, T system. These reduce to a force at (α, β), whose components parallel to the axes of x, y, and z will be taken to be

$$P_1 Bd\beta, \qquad U_1 Bd\beta, \qquad T_1 Bd\beta$$

respectively, and a couple whose components about the axes of x and y will be taken to be

$$H_1 Bd\beta, \qquad G_1 Bd\beta.$$

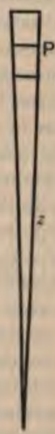
On the element of the face $\beta = \text{const.}$ whose trace on the middle-surface is of length $Ad\alpha$ there act forces of the U, Q, S system. These reduce to a force at (α, β), whose components parallel to the axes of x, y, and z will be taken to be

$$U_2 Ad\alpha, \qquad P_2 Ad\alpha, \qquad T_2 Ad\alpha$$

respectively, and a couple whose components about the axes of x and y will be taken to be

$$G_2 Ad\alpha, \qquad H_2 Ad\alpha.$$

In finding the values of these quantities in terms of the displacements we have to remember that the line-elements $Ad\alpha$ and $Bd\beta$ have finite curvature, so that in the normal plane through $Ad\alpha$ for example the element of area on which such a stress as Q acts is the product of dz and $Ad\alpha(1 - z/\rho_1)$ as shewn in the annexed figure.



The couples can be calculated as far as h^3, and we find

$$H_1 Bd\beta = \int_{-h}^{h} - Uz\, Bd\beta\, (1 - z/\rho_2)\, dz$$

and $$H_2 Ad\alpha = \int_{-h}^{h} Uz\, Ad\alpha\, (1 - z/\rho_1)\, dz.$$

From which, to a sufficient approximation,

$$H_1 = - H_2 = C(1 - \sigma)\,\tau = H \text{ say} \qquad \text{.........(39).}$$

Also
$$G_1 B d\beta = \int_{-h}^{h} Pz\, B d\beta\, (1 - z/\rho_2)\, dz,$$

$$G_2 A d\alpha = \int_{-h}^{h} - Qz\, A d\alpha\, (1 - z/\rho_1)\, dz.$$

From which to a sufficient approximation

$$G_1 = - C(\kappa_1 + \sigma\kappa_2),\quad G_2 = C(\kappa_2 + \sigma\kappa_1) \qquad \text{.........(40)};$$

in these expressions such quantities as ϵ_1/ρ_2 have been rejected as small compared with such quantities as κ_1.

Again we can calculate U_1 and U_2 as far as h^3; we have[1]

$$U_1 = \int_{-h}^{h} U(1 - z/\rho_2)\, dz,\quad U_2 = \int_{-h}^{h} U(1 - z/\rho_1)\, dz.$$

Hence

$$\left.\begin{aligned} U_1 &= \frac{3C(1-\sigma)}{2h^2}\varpi + C(1-\sigma)\frac{\tau}{\rho_2} - \tfrac{1}{2}C(1-\sigma)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\tau, \\ U_2 &= \frac{3C(1-\sigma)}{2h^2}\varpi + C(1-\sigma)\frac{\tau}{\rho_1} - \tfrac{1}{2}C(1-\sigma)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\tau \end{aligned}\right\} \text{...(41).}$$

Further in terms of the unknown R_0 we can calculate P_1 and P_2 as far as h^3; we have

$$P_1 = \int_{-h}^{h} P(1 - z/\rho_2)\, dz,\quad P_2 = \int_{-h}^{h} Q(1 - z/\rho_1)\, dz.$$

[1] Sufficiently exact values for the stress-couples are given by the first approximation. The necessity for determining the terms of the stress-resultants that contain C as well as those that contain C/h^2 was first noticed by Mr Basset in the particular cases of cylindrical and spherical shells (*Phil. Trans. R. S.* A. 1890). Mr Basset's investigation proceeds on entirely different lines from the present, and he retains terms in $C\epsilon_1$ as well as terms in $C\kappa_1$. The values for U_1, U_2 above, (his M_1, M_2,) are in substantial agreement with those given by him, the additional terms which he obtains being of the order which is here neglected. These terms could be obtained by the present process. The values found for P_1, P_2 above do not agree with those found by Mr Basset for the same stress-resultants (called by him T_1 and T_2), though the difference $P_1 - P_2$ is in substantial agreement with his difference $T_1 - T_2$. The discrepancy in the two accounts of the stress-system is in regard to the quantity R_0. By equating coefficients in two forms of the expression for the virtual work of a small displacement Mr Basset appears to have evaluated P_1 and P_2 completely in terms of displacements. We shall be able hereafter in discussing the two-dimensional vibrations of a complete circular cylindrical shell of infinite length to bring the matter to a test. It will appear that P_1 and P_2 can be found in that problem from the equations of small motion, and that the values which can thus be deduced are not identical with Mr Basset's values in terms of displacements.

Hence

$$\left.\begin{aligned} P_1 &= \frac{3C}{h^2}(\epsilon_1+\sigma\epsilon_2) + C\left[\frac{\kappa_1+\sigma\kappa_2}{\rho_2} - \left(\frac{\kappa_1}{\rho_1}+\frac{\sigma\kappa_2}{\rho_2}\right)\right. \\ &\qquad \left. -\tfrac{1}{2}\frac{\sigma}{1-\sigma}(\kappa_1+\kappa_2)\left(\frac{1}{\rho_1}+\frac{\sigma}{\rho_2}\right)+\tfrac{1}{2}\frac{\sigma R_0}{\mu}\right], \\ P_2 &= \frac{3C}{h^2}(\epsilon_2+\sigma\epsilon_1) + C\left[\frac{\kappa_2+\sigma\kappa_1}{\rho_1} - \left(\frac{\kappa_2}{\rho_2}+\frac{\sigma\kappa_1}{\rho_1}\right)\right. \\ &\qquad \left. -\tfrac{1}{2}\frac{\sigma}{1-\sigma}(\kappa_1+\kappa_2)\left(\frac{\sigma}{\rho_1}+\frac{1}{\rho_2}\right)+\tfrac{1}{2}\frac{\sigma R_0}{\mu}\right] \end{aligned}\right\} \quad \ldots(42).$$

Finally T_1 and T_2 cannot be calculated by this method.

340. General Equations of Equilibrium.

We have now to consider the equilibrium of an element of the shell bounded by normal sections through α, β, $\alpha+d\alpha$, $\beta+d\beta$, when subject to the action of forces $2Xh\,ABd\alpha d\beta$, $2Yh\,ABd\alpha d\beta$, $2Zh\,ABd\alpha d\beta$ parallel to the axes of x, y, z at (α, β), and couples $2Lh\,ABd\alpha d\beta$, $2Mh\,ABd\alpha d\beta$ about the axes of x and y.

In forming the equations of equilibrium of an element bounded by normal sections through lines of curvature we have to remember that the axes of reference undergo rotations as we pass from point to point of the surface. The rotations[1] executed by the axes of x, y, z about themselves as we pass from the face α to the face $\alpha+d\alpha$ are $p_1 d\alpha$, $q_1 d\alpha$, $r_1 d\alpha$; those executed as we pass from the face β to the face $\beta+d\beta$ are $p_2 d\beta$, $q_2 d\beta$, $r_2 d\beta$.

We can now put down the forces parallel to the axes that act on the four faces of the element.

On the face α we have

$$-P_1 Bd\beta \text{ parallel to } x,$$
$$-U_1 Bd\beta \text{ parallel to } y,$$
$$-T_1 Bd\beta \text{ parallel to } z;$$

on the face $\alpha+d\alpha$ we have

$$\left.\begin{aligned} &P_1 Bd\beta + d\alpha\frac{\partial}{\partial\alpha}(P_1 Bd\beta) - U_1 Bd\beta . r_1 d\alpha + T_1 Bd\beta . q_1 d\alpha, \\ &U_1 Bd\beta + d\alpha\frac{\partial}{\partial\alpha}(U_1 Bd\beta) - T_1 Bd\beta . p_1 d\alpha + P_1 Bd\beta . r_1 d\alpha, \\ &T_1 Bd\beta + d\alpha\frac{\partial}{\partial\alpha}(T_1 Bd\beta) - P_1 Bd\beta . q_1 d\alpha + U_1 Bd\beta . p_1 d\alpha \end{aligned}\right\} \ldots(43),$$

parallel to the same axes.

[1] In the calculations here given the displacement is supposed small; we shall presently examine the modification for finite displacements.

On the face β we have

$$-U_2 A d\alpha \text{ parallel to } x,$$
$$-P_2 A d\alpha \text{ parallel to } y,$$
$$-T_2 A d\alpha \text{ parallel to } z;$$

on the face $\beta + d\beta$ we have

$$\left.\begin{aligned} &U_2 A d\alpha + d\beta \frac{\partial}{\partial\beta}(U_2 A d\alpha) - P_2 A d\alpha \, . \, r_2 d\beta + T_2 A d\alpha \, . \, q_2 d\beta, \\ &P_2 A d\alpha + d\beta \frac{\partial}{\partial\beta}(P_2 A d\alpha) - T_2 A d\alpha \, . \, p_2 d\beta + U_2 A d\alpha \, . \, r_2 d\beta, \\ &T_2 A d\alpha + d\beta \frac{\partial}{\partial\beta}(T_2 A d\alpha) - U_2 A d\alpha \, . \, q_2 d\beta + P_2 A d\alpha \, . \, p_2 d\beta \end{aligned}\right\} \ldots (44),$$

parallel to the same axes.

Remembering the values of $p_1, q_1, \ldots$ given in art. 328, and equating to zero the sum of all the forces parallel to x, or y, or z that act on the element we obtain the three equations of resolution

$$\left.\begin{aligned} &\frac{1}{AB}\left[\frac{\partial(P_1 B)}{\partial\alpha} + \frac{\partial(U_2 A)}{\partial\beta} + U_1 \frac{\partial A}{\partial\beta} - P_2 \frac{\partial B}{\partial\alpha}\right] - \frac{T_1}{\rho_1} + 2hX = 0, \\ &\frac{1}{AB}\left[\frac{\partial(U_1 B)}{\partial\alpha} + \frac{\partial(P_2 A)}{\partial\beta} + U_2 \frac{\partial B}{\partial\alpha} - P_1 \frac{\partial A}{\partial\beta}\right] - \frac{T_2}{\rho_2} + 2hY = 0, \\ &\frac{1}{AB}\left[\frac{\partial(T_1 B)}{\partial\alpha} + \frac{\partial(T_2 A)}{\partial\beta}\right] + \frac{P_1}{\rho_1} + \frac{P_2}{\rho_2} + 2hZ = 0 \end{aligned}\right\} (45).$$

In order to form the equations of moments about the axes of x and y we write down the couples that act on the four faces.

On the face α we have

$$-H B d\beta \text{ about } x,$$
$$-G_1 B d\beta \text{ about } y;$$

on the face $\alpha + d\alpha$ we have

$$H B d\beta + d\alpha \frac{\partial}{\partial\alpha}(H B d\beta) - G_1 B d\beta \, . \, r_1 d\alpha,$$

$$G_1 B d\beta + d\alpha \frac{\partial}{\partial\alpha}(G_1 B d\beta) + H B d\beta \, . \, r_1 d\alpha,$$

about the same axes.

On the face β we have

$$-G_2 A d\alpha \text{ about } x,$$
$$H A d\alpha \text{ about } y;$$

on the face $\beta+d\beta$ we have

$$G_2 A d\alpha + d\beta \frac{\partial}{\partial\beta}(G_2 A d\alpha) + HA d\alpha \,.\, r_2 d\beta,$$

$$-HA d\alpha - d\beta \frac{\partial}{\partial\beta}(HA d\alpha) + G_2 A d\alpha \,.\, r_2 d\beta,$$

about the same axes.

The systems (43) and (44) give rise to moments

$$T_2 AB d\alpha d\beta \text{ and } -T_1 AB d\alpha d\beta$$

about the same axes.

Hence the equations of moments are

$$\left.\begin{aligned} &\frac{1}{AB}\left[\frac{\partial(HB)}{\partial\alpha} + \frac{\partial(G_2 A)}{\partial\beta} + H\frac{\partial B}{\partial\alpha} + G_1\frac{\partial A}{\partial\beta}\right] + T_2 + 2hL = 0, \\ &\frac{1}{AB}\left[\frac{\partial(G_1 B)}{\partial\alpha} - \frac{\partial(HA)}{\partial\beta} - H\frac{\partial A}{\partial\beta} + G_2\frac{\partial B}{\partial\alpha}\right] - T_1 + 2hM = 0 \end{aligned}\right\} \ldots\ldots (46),$$

where we have substituted for r_1, r_2 their values from (4) of art. 328.

Equations (45) and (46) are the general equations of equilibrium[1] when the displacements are small.

341. Equations for Finite Displacements. We shall hereafter, in connexion with the stability of certain configurations of equilibrium, find it important to have equations determining the stress-couples and the equations of equilibrium of a thin plate or shell held finitely deformed. For the general case of a thin shell deformed in such a way that the strained middle-surface is finitely different from, but applicable upon, the unstrained middle-surface we notice:

1°. That the lines $\alpha=$ const. and $\beta=$ const. which were lines of curvature do not in general remain lines of curvature, so that in general κ_1, κ_2, τ are all finite.

2°. That the stress-couples are $G_1 = -C(\kappa_1+\sigma\kappa_2), \ldots$ just as in the case of infinitesimal bending, but nothing has been proved as to the values of the stress-resultants. This result follows from the work in arts. 333—335, since in obtaining the expressions for the couples it is nowhere assumed that the displacement is infinitesimal.

[1] Except for the notation these equations are identical with those given by Prof. Lamb in *Proc. Lond. Math. Soc.* XXI. 1890. For cylindrical and spherical shells equivalent equations have been obtained by Mr Basset in *Phil. Trans. R. S.* (A), 1890.

3°. That the general equations of equilibrium can be obtained by the method of the preceding article, by writing in all such expressions as (43) $p_1', q_1', r_1', p_2', \ldots$ in place of $p_1, \ldots$ *i.e.* by referring to the strained surface instead of the unstrained surface.

We shall now shew how to write down the equations in two particular cases of great importance. The first of these is the case where the lines of curvature remain lines of curvature. The quantities r_1, r_2 are unaltered by the strain; the quantities p_1, q_2 vanish after strain as well as before strain; and thus τ vanishes, and κ_1 and κ_2 are $1/\rho_1' - 1/\rho_1$ and $1/\rho_2' - 1/\rho_2$, where ρ_1', ρ_2' are the principal radii of curvature of the strained middle-surface. The equations of equilibrium are to be obtained by writing ρ_1' and ρ_2' in place of ρ_1 and ρ_2 in (45), and by putting

$$\left.\begin{aligned} G_1 &= -C\left[(1/\rho_1' - 1/\rho_1) + \sigma(1/\rho_2' - 1/\rho_2)\right], \\ G_2 &= C\left[(1/\rho_2' - 1/\rho_2) + \sigma(1/\rho_1' - 1/\rho_1)\right], \\ H &= 0 \end{aligned}\right\}$$

in (46).

The second particular case is that of a naturally plane plate referred to lines $\alpha = \text{const.}$ and $\beta = \text{const.}$, which in the unstrained state are parallel to Cartesian rectangular axes drawn on the middle-surface. These lines do not in general become lines of curvature of the strained middle-surface. The latter surface is a developable, and in the applications that we shall have to make it may be treated as given, so that κ_1, κ_2 and τ may be supposed given. The stress-couples are

$$G_1 = -C(\kappa_1 + \sigma\kappa_2), \quad G_2 = C(\kappa_2 + \sigma\kappa_1), \quad H = C\tau,$$

and the equations of equilibrium are

$$\left.\begin{aligned} &\frac{\partial P_1}{\partial\alpha} + \frac{\partial U_2}{\partial\beta} - T_1\kappa_1 - T_2\tau + 2hX = 0, \\ &\frac{\partial U_1}{\partial\alpha} + \frac{\partial P_2}{\partial\beta} - T_1\tau - T_2\kappa_2 + 2hY = 0, \\ &\frac{\partial T_1}{\partial\alpha} + \frac{\partial T_2}{\partial\beta} + P_1\kappa_1 + P_2\kappa_2 + (U_1 + U_2)\tau + 2hZ = 0, \end{aligned}\right\}$$

and

$$\left.\begin{aligned} &\frac{\partial H}{\partial\alpha} + \frac{\partial G_2}{\partial\beta} + T_2 + 2hL = 0, \\ &\frac{\partial G_1}{\partial\alpha} - \frac{\partial H}{\partial\beta} - T_1 + 2hM = 0. \end{aligned}\right\}$$

We shall not at present make any further investigation of finite deformations.

342. Boundary-conditions.

Let P_1', U_1', T_1' be the stress-resultants at any point of the edge-line, P_1' being along the normal to the edge-line drawn on the middle-surface, U_1' along the tangent to the edge-line, and T_1' along the normal to the middle-surface. Also let H', G' be the stress-couples on an element of the edge, H' being the torsional couple in the plane of the edge, and G' the flexural couple about the tangent to the edge-line. All these are estimated per unit of length of the edge-line.

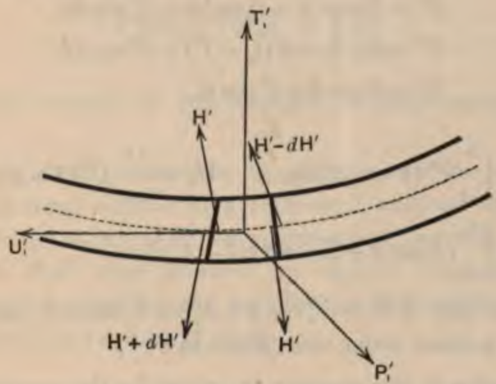


Fig. 54.

This system has to be replaced by a simpler statically equivalent system by replacing the couple $H'ds$ by forces in the plane of the edge.

Let ρ' be the radius of curvature of the normal section through the tangent to the edge-line. Then it is clear from the figure that the equivalent distribution of force gives us a force $-\partial H'/\partial s$ to be added to T_1', and a force H'/ρ' to be added to U_1'.[1]

Hence if $\mathfrak{P}$, $\mathfrak{U}$, $\mathfrak{T}$, $\mathfrak{G}$, $\mathfrak{H}$ are the applied forces and couples the boundary-conditions are

$$\left.\begin{aligned} P_1' &= \mathfrak{P}, & U_1' + H'/\rho' &= \mathfrak{U} + \mathfrak{H}/\rho', \\ G' &= \mathfrak{G}, & T_1' - \frac{\partial H'}{\partial s} &= \mathfrak{T} - \frac{\partial \mathfrak{H}}{\partial s} \end{aligned}\right\} \quad \ldots\ldots\ldots(47).$$

[1] This was remarked by Prof. Lamb and Mr Basset in the papers quoted on p. 241.

In all the particular cases that we shall investigate the boundary of the shell will be composed of normal sections through lines of curvature. When this is the case we can proceed to form the expressions for P_1'... without difficulty. If however the normal to the edge-line make an angle θ with the line of curvature $\beta = \text{const.}$, greater care is required in the evaluation of the stress-resultants P_1'.... Just as in art. 310 this may be done by first expressing the internal stresses P'... across an element of area of the edge. Equations (15) of that article hold without modification, so that we have

$$P' = P\cos^2\theta + Q\sin^2\theta + U\sin 2\theta,$$
$$U' = \sin\theta\cos\theta\,(Q-P) + U\cos 2\theta,$$
$$T' = S\sin\theta + T\cos\theta.$$

Also

$$P_1' = \int_{-h}^{h} P'(1 - z/\rho')\,dz, \qquad U_1' = \int_{-h}^{h} U'(1 - z/\rho')\,dz,$$

$$T_1' = \int_{-h}^{h} (S\sin\theta + T\cos\theta)(1 - z/\rho')\,dz,$$

of which the last is $T_1' = T_1\cos\theta + T_2\sin\theta$, since S and T contain no terms of a lower order than those in z^2.

The couples in like manner are given by the equations

$$H' = \int_{-h}^{h} -U'z(1 - z/\rho')\,dz,$$

$$G' = \int_{-h}^{h} P'z(1 - z/\rho')\,dz.$$

To complete the formulæ we have only to add Euler's formula

$$1/\rho' = \sin^2\theta/\rho_1 + \cos^2\theta/\rho_2,$$

which expresses the curvature of the normal section through the edge-line in terms of the curvatures of the normal sections through the lines of curvature.

CHAPTER XXII.

APPLICATIONS OF THE THEORY OF THIN SHELLS.

343. WE shall consider in the first place the form of the equations of small motion of a thin shell, and we shall see that from the form alone some very important conclusions can be drawn. We shall then proceed, by way of illustration of the general theory, to consider the equilibrium and small vibrations of thin cylindrical and spherical shells.

344. Equations of free vibration.

The general equations of small vibration are deduced from the equations of equilibrium (45) and (46) of art. 340 by replacing the forces and couples X, Y, Z, and L, M by the kinetic reactions against acceleration. If we reject "rotatory inertia" the couples L, M vanish, and the forces X, Y, Z have to be replaced by $-\rho\dfrac{\partial^2 \mathrm{u}}{\partial t^2}$, $-\rho\dfrac{\partial^2 \mathrm{v}}{\partial t^2}$, $-\rho\dfrac{\partial^2 \mathrm{w}}{\partial t^2}$, where u, v, w are the displacements of a point of the middle-surface in the directions of the tangents to the lines of curvature $\beta=$ const., $\alpha=$ const., and the normal, and ρ is the density of the material.

The equations of moments (46) of art. 340 give us

$$\left.\begin{aligned} T_2 &= -\frac{1}{AB}\left[\frac{\partial(HB)}{\partial\alpha}+\frac{\partial(G_2A)}{\partial\beta}+H\frac{\partial B}{\partial\alpha}+G_1\frac{\partial A}{\partial\beta}\right], \\ T_1 &= \frac{1}{AB}\left[\frac{\partial(G_1B)}{\partial\alpha}-\frac{\partial(HA)}{\partial\beta}-H\frac{\partial A}{\partial\beta}+G_2\frac{\partial B}{\partial\alpha}\right] \end{aligned}\right\}\ldots(1);$$

and the equations of resolution (45) of the same article give us

$$\left.\begin{aligned}
2\rho h\frac{\partial^2 \mathrm{u}}{\partial t^2} &= \frac{1}{AB}\left[\frac{\partial(P_1B)}{\partial\alpha}+\frac{\partial(U_2A)}{\partial\beta}+U_1\frac{\partial A}{\partial\beta}-P_2\frac{\partial B}{\partial\alpha}\right]-\frac{T_1}{\rho_1},\\
2\rho h\frac{\partial^2 \mathrm{v}}{\partial t^2} &= \frac{1}{AB}\left[\frac{\partial(U_1B)}{\partial\alpha}+\frac{\partial(P_2A)}{\partial\beta}+U_2\frac{\partial B}{\partial\alpha}-P_1\frac{\partial A}{\partial\beta}\right]-\frac{T_2}{\rho_2},\\
2\rho h\frac{\partial^2 \mathrm{w}}{\partial t^2} &= \frac{1}{AB}\left[\frac{\partial(T_1B)}{\partial\alpha}+\frac{\partial(T_2A)}{\partial\beta}\right]+\frac{P_1}{\rho_1}+\frac{P_2}{\rho_2}
\end{aligned}\right\}\ldots(2).$$

If we substitute for T_1 and T_2 from (1) in (2) we shall have the general equations.

The stress-resultants P_1, P_2, U_1, U_2 and the stress-couples G_1, G_2, H are connected with the quantities ϵ_1, ϵ_2, ϖ, that define the extension, and κ_1, κ_2, τ, that define the flexure by the formulæ (39), (40), (41) and (42) of art. 339, while the quantities ϵ_1, ϵ_2, ϖ and κ_1, κ_2, τ are given in terms of the displacements by the formulæ (18) of art. 331 and (25) of art. 332.

Now an inspection of these formulæ shews that P_1, P_2, U_1, U_2 each consists of two parts. The first part is the product of C/h^2 and a linear function of ϵ_1, ϵ_2, ϖ; the second is the product of C and a linear function of κ_1, κ_2, τ, and an additional quantity R_0 whose expression in terms of the displacements is unknown[1]. The expressions for G_1, G_2, H contain in like manner the cylindrical rigidity C and linear functions of κ_1, κ_2, τ. When the shell is indefinitely thin the two parts of the expressions for the stress-resultants may be expected to be of different orders of magnitude, and we may expect that one or other of them ought to be rejected.

Suppose in fact that any normal vibration is being executed, then u, v, w will be the products of a small constant coefficient defining the amplitude of the vibration, a simple harmonic function of the time, and certain functions of α, β. In general we cannot from analytical considerations predict that any of the quantities ϵ_1, ϵ_2, ϖ, κ_1, κ_2, τ will be small in comparison with any of the others. From analytical considerations alone we should therefore expect that the terms in C/h^2 which depend on extension are to be retained, while the terms in C which depend upon flexure are to be rejected.

[1] R_0 can be found in some particular cases by solution of the equations of vibration. When the type of vibration is completely determined R_0 will be known.

345. Are the vibrations extensional?

Let us develope briefly the consequences of this position. It is in the first place contrary to the prevalent impression that the vibrations of a bell depend upon bending—an impression which in the early stages of the subject was regarded as a guiding axiom. Bending accompanies the deformation, and is of the same order of magnitude as the extension by which the deformation is characterised, but the stresses that depend upon such bending are indefinitely small in comparison with those that depend upon the comparable stretching of the middle-surface. If it could be shewn that the equations could be satisfied by displacements u, v, w involving stretching to the extent suggested, and that the modes of vibration thence deducible included all possible modes, or all modes of practical importance, it would be unnecessary to proceed with the theory except in so far as these extensional modes are concerned. If however it can be shewn that any important mode of vibration is not included, we shall have to seek for some different theory. The crucial test of the theory will be found in the way in which the thickness h enters into the expression for the frequency.

Now the equations of extensional vibration are

$$\left.\begin{aligned} 2\rho h\frac{\partial^2 \mathrm{u}}{\partial t^2} &= \frac{1}{AB}\left[\frac{\partial(P_1B)}{\partial\alpha}+\frac{\partial(U_2A)}{\partial\beta}+U_1\frac{\partial A}{\partial\beta}-P_2\frac{\partial B}{\partial\alpha}\right],\\ 2\rho h\frac{\partial^2 \mathrm{v}}{\partial t^2} &= \frac{1}{AB}\left[\frac{\partial(U_1B)}{\partial\alpha}+\frac{\partial(P_2A)}{\partial\beta}+U_2\frac{\partial B}{\partial\alpha}-P_1\frac{\partial A}{\partial\beta}\right],\\ 2\rho h\frac{\partial^2 \mathrm{w}}{\partial t^2} &= \frac{P_1}{\rho_1}+\frac{P_2}{\rho_2}\end{aligned}\right\}\quad\ldots(3),$$

in which

$$P_1=\frac{3C}{h^2}(\epsilon_1+\sigma\epsilon_2),\qquad P_2=\frac{3C}{h^2}(\epsilon_2+\sigma\epsilon_1),\qquad U_1=U_2=\frac{3C(1-\sigma)}{2h^2}\varpi\qquad\ldots\ldots\ldots\ldots(4),$$

and

$$\left.\begin{aligned}\epsilon_1 &= \frac{1}{A}\frac{\partial\mathrm{u}}{\partial\alpha}+\frac{\mathrm{v}}{AB}\frac{\partial A}{\partial\beta}-\frac{\mathrm{w}}{\rho_1},\\ \epsilon_2 &= \frac{1}{B}\frac{\partial\mathrm{v}}{\partial\beta}+\frac{\mathrm{u}}{AB}\frac{\partial B}{\partial\alpha}-\frac{\mathrm{w}}{\rho_2},\\ \varpi &= \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{\mathrm{v}}{B}\right)+\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{\mathrm{u}}{A}\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and the conditions that hold at a free edge $\alpha=$ const. are $P_1=0$ and $U_1=0$ or $\epsilon_1+\sigma\epsilon_2=0$, $\varpi=0$.

If we substitute for P_1, P_2, U_1, U_2 in (3) their values in terms of displacements, and suppose u, v, w proportional to $\epsilon^{\iota pt}$, so that $p/2\pi$ is the frequency of the vibration, we shall obtain equations which can be written in the form

$$2\rho h p^2 \mathrm{u} = C/h^2 \times (\text{a function of u, v, w}).$$

These equations can be solved[1], and the boundary-conditions can be satisfied, and there will be an equation giving $\rho p^2 h^3/C$ in terms of quantities independent of h. Since C is

$$\tfrac{8}{3}\mu h^3 (\lambda+\mu)/(\lambda+2\mu),$$

this equation gives values of p^2 independent of h, so that the frequency is independent of the thickness of the shell.

346. Necessary existence of modes mainly non-extensional.

We may now see, by means of general dynamical principles, that the extensional modes do not include all the modes of vibration of the thin shell[2]. We have given in art. 338 an expression for the potential energy of the strained shell, and we may write down an expression for the kinetic energy per unit area in the form

$$\rho h \left[\left(\frac{\partial \mathrm{u}}{\partial t}\right)^2 + \left(\frac{\partial \mathrm{v}}{\partial t}\right)^2 + \left(\frac{\partial \mathrm{w}}{\partial t}\right)^2\right] \text{................} (6),$$

rotatory inertia being neglected. If we assume any forms whatever for u, v, w, and substitute in the kinetic and potential energies, we shall be able to obtain an expression for the frequency of the shell when constrained to vibrate according to the assumed type; and it is a general dynamical principle that the frequency obtained by assuming the type cannot be less than the least frequency natural to the system[3]. The principle is independent of any approximation of the assumed to a possible type.

In particular we may assume that the type is such that no line on the middle-surface is altered in length[4], or that the middle-

[1] Particular cases will be worked out below.

[2] See Lord Rayleigh 'On the Bending and Vibrations of thin Elastic Shells especially of Cylindrical Form'. *Proc. R. S.* XLV. 1889.

[3] See Lord Rayleigh's *Theory of Sound*, vol. I. art. 89.

[4] If any other assumption be made the assumed type is extensional and the frequency is independent of the thickness.

surface moves so as always to be applicable upon itself. In this case the potential energy per unit area is

$$\tfrac{1}{2}C\left[(\kappa_1+\kappa_2)^2-2(1-\sigma)(\kappa_1\kappa_2-\tau^2)\right]\ldots\ldots\ldots\ldots\ldots(7),$$

where κ_1, κ_2, τ are definite linear functions of u, v, w and their differential coefficients given by (25) of art. 332. The conditions that no line on the middle-surface is altered in length are that ϵ_1, ϵ_2, and ϖ are zero, or by (18) of art. 331,

$$\left.\begin{aligned}\frac{1}{A}\frac{\partial \mathrm{u}}{\partial\alpha}+\frac{\mathrm{v}}{AB}\frac{\partial A}{\partial\beta}-\frac{\mathrm{w}}{\rho_1}=0,\\ \frac{1}{B}\frac{\partial \mathrm{v}}{\partial\beta}+\frac{\mathrm{u}}{AB}\frac{\partial B}{\partial\alpha}-\frac{\mathrm{w}}{\rho_2}=0,\\ \frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{\mathrm{v}}{B}\right)+\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{\mathrm{u}}{A}\right)=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

From the first two of these equations we obtain

$$\frac{\rho_1}{A}\left(\frac{\partial \mathrm{u}}{\partial\alpha}+\frac{\mathrm{v}}{B}\frac{\partial A}{\partial\beta}\right)=\frac{\rho_2}{B}\left(\frac{\partial \mathrm{v}}{\partial\beta}+\frac{\mathrm{u}}{A}\frac{\partial B}{\partial\alpha}\right),$$

which, with the third of the same set of equations, forms a system of two simultaneous linear partial differential equations of the first order for the determination of u, v. When u, v are found from these equations w is determined, and thus we see that the conditions of inextensibility lead to certain forms for u, v, w.

There exist therefore systems of displacements which satisfy the condition that no line on the middle-surface is altered in length. When these are substituted in the kinetic and potential energies, and the frequency deduced, we find a relation of the form

$$p^2 \propto C/h \propto h^2,$$

so that the frequency is proportional to the thickness. By taking the shell sufficiently thin we may make the frequency corresponding to the assumed type as small as we please in comparison with that corresponding to any extensional mode. We must therefore conclude that there are modes of vibration much graver than the extensional ones.

The above argument is quite independent of any physical considerations as to the ease with which a shell can be bent as compared with the difficulty of producing extension, but such considerations shew the importance of the non-extensional modes proved to exist.

347. Are the vibrations purely non-extensional?

Let us now consider how it may be possible to satisfy the general equations when the terms in C cannot be rejected in comparison with those in C/h^2. When this is the case it is clear that, in order that the equations may hold, ϵ_1, ϵ_2, ϖ must be small of the order $\kappa_1 h^2$ at least, *i.e.* the extension must be of the order of the square of the thickness multiplied by the change of curvature, or it may be small of a higher order. In either case an *approximation* to the solution of the problem will be found by supposing the quantities defining the extension to vanish identically. Thus, whether the equations can be satisfied by displacements involving no extension of the middle-surface or not, such displacements always constitute a close approximation to the actual displacements in any of the graver modes of vibration.

This conclusion is very important. From the assumed type we learnt only a superior limit to the order of magnitude of the frequencies of the gravest tones. We now infer that the assumed type is a close approximation to the actual type, and we know[1] that the correction to the period for the departure of the actual from the assumed type of motion is always a small quantity of the second order when the correction to the value of any displacement is regarded as of the first order. Thus the type is approximately defined by the kinematical conditions of inextensibility, and the period is actually deducible from them.

It is an important question whether the actual type can coincide with the assumed, in other words whether free vibrations of the shell can take place in which no line on the middle-surface is altered in length. A simple enumeration of the equations and the unknowns is sufficient to shew that in general this is not the case. Let us suppose that the displacements u, v, w satisfy not only the general equations obtained from (2) by substituting for T_1 and T_2 from (1) but also the conditions of inextensibility (8). We shall have really four quantities u, v, w, R_0 connected by six equations. We could in two ways eliminate u, v, w, R_0, and we should obtain two values for p^2 generally different from each other and from the value obtained by the energy method. But if we had substituted in the equations of small motion a set of values

[1] Lord Rayleigh's *Theory of Sound*, vol. I. art. 90.

for the displacements which are possible normal functions, the equations would have been compatible[1] and would have led to the same value for p^2. We must expect then that in ordinary circumstances the equations of small motion cannot be satisfied exactly by displacements for which the middle-surface is unextended, but that there may be exceptional circumstances in which they can be so satisfied. We shall examine later cases of both kinds.

348. Subsidiary displacements required to satisfy the differential equations.

Now seeing that the middle-surface cannot remain accurately unextended, and that the extension that occurs must be small of the order (square of thickness) × (change of curvature), we shall find the following method of approximate solution :—

We seek first the general forms of u, v, w which satisfy the equations (8) of inextensibility, then we substitute in the kinetic and potential energies, and determine the approximate forms of the normal functions and the period. Let u_0, v_0, w_0 be a set of values of u, v, w which correspond to a normal type of vibration when every line on the middle-surface is unaltered in length, and $p/2\pi$ the corresponding frequency. We know that p is proportional to the thickness h.

Let the actual displacements be given by the equations

$$u = u_0 + h^2 u', \quad v = v_0 + h^2 v', \quad w = w_0 + h^2 w' \ldots\ldots(9),$$

where u′, v′, w′ are functions of α, β to be determined, and are comparable with u_0, v_0, w_0. Then in the terms of P_1, P_2,... which contain C/h^2 we may omit u_0, v_0, w_0 since they contribute nothing to ϵ_1, ϵ_2, and ϖ, and in the terms which contain C or p^2 we may omit h^2u', h^2v', h^2w'. Remembering that C is proportional to h^3, and that p is proportional to h, the quantity h will be seen to disappear from the equations, and we shall have three differential equations to determine four quantities u′, v′, w′ and R_0. From

[1] For example consider the vibrations of a string. The displacement y satisfies the equation $d^2y/dt^2 = a^2 d^2y/dx^2$. If we substitute for y a value proportional to $\sin nx$ we shall obtain the correct value for the frequency $p/2\pi$, viz.: we find $p^2 = n^2a^2$. Generally, if possible normal functions be substituted in the equations of small motion of any system, those equations determine the period without reference to the boundary-conditions.

the indeterminateness hence arising it appears that there is an infinity of modes differing very little from the corresponding non-extensional mode, although generally none of them are identical with it. We shall give later an example of the determination as far as is possible of u′, v′, w′.

Now until u′, v′, w′ are determined ϵ_1, ϵ_2, and ϖ will be unknown, and therefore P_1, P_2, and U_1 may be regarded as unknown, while $U_1 - U_2$ is known. In fact U_1 and U_2 contain the same unknown quantity $\frac{2}{3}C(1-\sigma)\varpi/h^2$, and their remaining terms are known functions of τ. We have therefore in the first instance three differential equations to determine P_1, P_2, and ϖ, and when these are solved in accordance with the boundary-conditions we may proceed to substitute for P_1 and P_2 from (42) of art. 339, and so obtain two equations connecting ϵ_1 and ϵ_2 with the unknown R_0.

349. Subsidiary displacements required to satisfy the boundary-conditions.

It remains to examine how the approximately non-extensional modes can satisfy the boundary-conditions at a free edge, and for this purpose we shall consider the case where a line of curvature $\alpha=$ const. of the unstrained middle-surface is the edge-line. The boundary-conditions of art. 342 become

$$\left.\begin{array}{ll} P_1 = 0, & U_1 + H/\rho_2 = 0, \\ G_1 = 0, & T_1 - \dfrac{1}{B}\dfrac{\partial H}{\partial \beta} = 0 \end{array}\right\} \text{..............}(10),$$

where T_1 is determined by the second of (1).

With the reductions made as above in the differential equations it will be found that P_1, P_2, U_1, and U_2 are all determinate from the differential equations and the first two boundary-conditions.

If we suppose that the specification (9) holds right up to the boundary then the third condition $G_1 = 0$ is $C(\kappa_1 + \sigma\kappa_2) = 0$, and we shall find on examining particular cases that this cannot be satisfied by the values of κ_1, κ_2 deduced from the expressions u_0, v_0, w_0 as displacements. We must therefore conclude that near the boundary either the subsidiary displacements which must be superposed upon u_0, v_0, w_0, or some of their differential coefficients which

occur in κ_1, κ_2 and τ, cease to be of the same order of magnitude as $h^2\mathrm{u}_0$, $h^2\mathrm{v}_0$, $h^2\mathrm{w}_0$ and their corresponding differential coefficients.

Generally we ought not to expect that the condition $G_1=0$ could be satisfied by non-extensional displacements; for such displacements are determined independently of any such condition. Even if the equation $G_1=0$ were satisfied at the edge, there would remain the condition $T_1-\dfrac{1}{B}\dfrac{\partial H}{\partial \beta}=0$, and from the form of the equations of inextensibility it is clear that two boundary-conditions could not in general be satisfied[1]. From what we have said it appears that without some modification the solutions of the equations that express the condition that no line on the middle-surface is altered in length are incapable by themselves of satisfying either the differential equations of free vibration or the boundary-conditions at a free edge. The discrepancy thus arising can only be explained by taking account of the extension of the middle-surface.

The difficulty of the analysis seems to preclude the possibility of a verification by the complete investigation of any particular case of vibrations, but the solution of a certain statical problem[2] gives an idea of the way in which subsidiary displacements may be found which give rise to terms of the order required.

Suppose a is a finite linear dimension of the shell (*e.g.* the radius when the shell is cylindrical), and consider a displacement to be added to w_0 which is comparable with

$$\mathrm{w}_0\frac{h}{a}e^{-q(\alpha_0-\alpha)},$$

where α_0 is the value of α at the free edge, and q is comparable with $\sqrt{(a/h)}$, and suppose the parameter α to diminish as we pass inwards along the shell from the edge. Since ϵ_1 contains a term

[1] This is the argument upon which I proceeded in my paper (*Phil. Trans.* A. 1888). Lord Rayleigh in his paper (*Proc. R. S.* XLV. 1889) contends that the boundary-conditions are satisfied by the non-extensional solutions. Referring to one of my equations (virtually the same as $P_1=0$), he notices that the terms in it which contain h as a factor are linear in ϵ_1 and ϵ_2, and the remaining terms contain h^3 as a factor, and he holds that, in consequence of the smallness of h, such an equation is sufficiently satisfied by the vanishing of ϵ_1 and ϵ_2. This argument does not apply to an equation like $G_1=0$, all whose terms contain the same power of h.

[2] Given by Prof. Lamb, *Proc. Lond. Math. Soc.* XXI. 1890.

$-\mathrm{w}/\rho_1$, such a displacement gives rise to extensional strains which, near the free edge, are comparable with the non-extensional strains $h\kappa_1$, $h\kappa_2$, $h\tau$ calculated from u_0, v_0, w_0, and at a little distance from the free edge are quite insensible. At the same time, since κ_1 contains a term $\frac{1}{A^2}\frac{\partial^2 \mathrm{w}}{\partial\alpha^2}$, such a displacement gives rise to changes of curvature which, near the free edge, are comparable with those arising from u_0, v_0, w_0. Terms of this kind are therefore adapted for satisfying at the same time such boundary-conditions as $P_1 = 0$ and such boundary-conditions as $G_1 = 0$.

We conclude that for the complete satisfaction of the differential equations and boundary-conditions there are required two sets of subsidiary displacements. The first set are everywhere of the order $h^2\mathrm{w}_0/a^2$, and they enable us to satisfy the differential equations; the second set diminish in geometric progression as we pass inwards from a free edge, but close to the edges they rise in importance so as to secure the satisfaction of the boundary-conditions. The extensional strains proved to exist are practically confined to a narrow region near the free edges; they give rise to stress-resultants which in general are of the same order as those on which the bending depends but near the edges tend to become very much greater; the stress-couples to which they give rise are of importance only in the neighbourhood of the edges.

Note. It is worthy of notice that the difficulties encountered in the theory of thin shells, and explained by taking account of the extension, do not occur in the corresponding theory of thin wires. In the latter theory the equations of vibration, the condition of inextensibility, and the terminal conditions form a system of equations which can be satisfied.

CYLINDRICAL SHELLS.

350. General equations.

We shall proceed to consider the case of a thin shell whose middle-surface when unstrained is a circular cylinder of radius a.

Let P be any point of the middle-surface, and let x be the distance of P from a fixed circular section measured along the generator through P, and let ϕ be the angle which the plane through P and the axis makes with a fixed plane through the axis; then x and ϕ are the parameters of the lines of curvature. If we take $\alpha = x$ and $\beta = \phi$ we shall have, in the notation of art. 328,

$$A = 1, \quad B = a, \quad 1/\rho_1 = 0, \quad 1/\rho_2 = -1/a \ldots\ldots(11)^1.$$

Let the displacements of P be u along the generator, v along the circular section, and w along the normal drawn outwards; then equations (18) of art. 331 become

$$\epsilon_1 = \frac{\partial \text{u}}{\partial x}, \quad \epsilon_2 = \frac{1}{a}\frac{\partial \text{v}}{\partial \phi} + \frac{\text{w}}{a}, \quad \varpi = \frac{\partial \text{v}}{\partial x} + \frac{1}{a}\frac{\partial \text{u}}{\partial \phi} \ldots\ldots(12),$$

and equations (25) of art. 332 give us

$$\kappa_1 = \frac{\partial^2 \text{w}}{\partial x^2}, \quad \kappa_2 = \frac{1}{a^2}\left(\frac{\partial^2 \text{w}}{\partial \phi^2} - \frac{\partial \text{v}}{\partial \phi}\right), \quad \tau = \frac{1}{a}\left(\frac{\partial^2 \text{w}}{\partial x \partial \phi} - \frac{\partial \text{v}}{\partial x}\right) \ldots(13).$$

The general equations of equilibrium under forces X, Y, Z per unit volume parallel to the generator, the tangent to the circular section, and the normal drawn outwards, are[2]

$$\left.\begin{aligned}
-2hX &= \frac{\partial P_1}{\partial x} + \frac{1}{a}\frac{\partial U_2}{\partial \phi}, \\
-2hY &= \frac{\partial U_1}{\partial x} + \frac{1}{a}\frac{\partial P_2}{\partial \phi} - \frac{1}{a}\left[\frac{\partial H}{\partial x} + \frac{1}{a}\frac{\partial G_2}{\partial \phi}\right], \\
-2hZ &= \frac{\partial}{\partial x}\left[\frac{\partial G_1}{\partial x} - \frac{1}{a}\frac{\partial H}{\partial \phi}\right] - \frac{1}{a}\frac{\partial}{\partial \phi}\left[\frac{\partial H}{\partial x} + \frac{1}{a}\frac{\partial G_2}{\partial \phi}\right] - \frac{P_2}{a}
\end{aligned}\right\} \ldots(14).$$

[1] The negative sign is taken because the radius of curvature ρ_2 is positive when the centre of curvature is reached by going from the surface in the positive direction of the axis z of art. 329, and this direction is that of the normal drawn outwards.

[2] It may be as well to remark that P_1 is a tension across the circular section x=const. in the direction x increasing, P_2 is a tension across the generator ϕ=const. in the direction ϕ increasing, U_1 is a tangential stress across the circular section, U_2 a tangential stress across the generator, G_1 is a flexural stress-couple acting across the circular section, G_2 a flexural stress-couple acting across the generator, and H is a torsional couple. Equal torsional couples H act across the generators and the circular sections.

and the equations of small motion are derived from these by substituting

$$-\rho\frac{\partial^2 \mathrm{u}}{\partial t^2} \text{ for } X, \quad -\rho\frac{\partial^2 \mathrm{v}}{\partial t^2} \text{ for } Y, \quad \text{and } -\rho\frac{\partial^2 \mathrm{w}}{\partial t^2} \text{ for } Z.$$

Further in the above equations

$$\left.\begin{aligned}
P_1 &= \frac{3C}{h^2}(\epsilon_1+\sigma\epsilon_2)+C\left[-\frac{\kappa_1}{a}+\tfrac{1}{2}\frac{\sigma^2}{1-\sigma}\frac{\kappa_1+\kappa_2}{a}+\tfrac{1}{2}\frac{\sigma R_0}{\mu}\right],\\
P_2 &= \frac{3C}{h^2}(\epsilon_2+\sigma\epsilon_1)+C\left[\frac{\kappa_2}{a}+\tfrac{1}{2}\frac{\sigma}{1-\sigma}\frac{\kappa_1+\kappa_2}{a}+\tfrac{1}{2}\frac{\sigma R_0}{\mu}\right],\\
U_1 &= \frac{3C(1-\sigma)}{2h^2}\varpi-\tfrac{1}{2}C(1-\sigma)\frac{\tau}{a},\\
U_2 &= \frac{3C(1-\sigma)}{2h^2}\varpi+\tfrac{1}{2}C(1-\sigma)\frac{\tau}{a},\\
G_1 &= -C(\kappa_1+\sigma\kappa_2),\ G_2=C(\kappa_2+\sigma\kappa_1),\ H=C(1-\sigma)\tau
\end{aligned}\right\}\ldots(15).$$

351. Extensional Vibrations.

Unless ϵ_1, ϵ_2, ϖ are all very approximately equal to zero, the terms that contain them are the most important, and we find the equations of extensional vibration in the form:

$$\left.\begin{aligned}
2\rho h\frac{\partial^2 \mathrm{u}}{\partial t^2} &= \frac{3C}{h^2}\left\{\frac{\partial}{\partial x}\left[\frac{\partial \mathrm{u}}{\partial x}+\frac{\sigma}{a}\left(\frac{\partial \mathrm{v}}{\partial \phi}+\mathrm{w}\right)\right]\right.\\
&\qquad\left.+\tfrac{1}{2}(1-\sigma)\frac{1}{a}\frac{\partial}{\partial \phi}\left[\frac{\partial \mathrm{v}}{\partial x}+\frac{1}{a}\frac{\partial \mathrm{u}}{\partial \phi}\right]\right\},\\
2\rho h\frac{\partial^2 \mathrm{v}}{\partial t^2} &= \frac{3C}{h^2}\left\{\tfrac{1}{2}(1-\sigma)\frac{\partial}{\partial x}\left[\frac{\partial \mathrm{v}}{\partial x}+\frac{1}{a}\frac{\partial \mathrm{u}}{\partial \phi}\right]\right.\\
&\qquad\left.+\frac{1}{a}\frac{\partial}{\partial \phi}\left[\frac{1}{a}\left(\frac{\partial \mathrm{v}}{\partial \phi}+\mathrm{w}\right)+\sigma\frac{\partial \mathrm{u}}{\partial x}\right]\right\},\\
2\rho h\frac{\partial^2 \mathrm{w}}{\partial t^2} &= -\frac{3C}{h^2}\left\{\frac{1}{a}\left[\sigma\frac{\partial \mathrm{u}}{\partial x}+\frac{1}{a}\left(\frac{\partial \mathrm{v}}{\partial \phi}+\mathrm{w}\right)\right]\right\}
\end{aligned}\right\}\ldots\ldots\ldots(16).$$

Suppose that as functions of the time u, v, w are proportional to $e^{\iota pt}$, and write

$$\kappa^2/a^2=\tfrac{2}{3}\rho h^3 p^2/C\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)^1;$$

[1] It is to be remembered that $C=\frac{4}{3}\mu h^3\frac{\lambda+\mu}{\lambda+2\mu}=\frac{2}{3}\frac{\mu h^3}{1-\sigma}$.

then the equations become

$$\left.\begin{aligned}
&\frac{\partial^2 \mathrm{u}}{\partial x^2}+\frac{\kappa^2}{a^2}\mathrm{u}+\tfrac{1}{2}(1-\sigma)\frac{1}{a^2}\frac{\partial^2\mathrm{u}}{\partial\phi^2}+\tfrac{1}{2}(1+\sigma)\frac{1}{a}\frac{\partial^2\mathrm{v}}{\partial x\partial\phi}+\frac{\sigma}{a}\frac{\partial\mathrm{w}}{\partial x}=0,\\
&\frac{1}{a^2}\frac{\partial^2\mathrm{v}}{\partial\phi^2}+\frac{\kappa^2}{a^2}\mathrm{v}+\tfrac{1}{2}(1-\sigma)\frac{\partial^2\mathrm{v}}{\partial x^2}+\tfrac{1}{2}(1+\sigma)\frac{1}{a}\frac{\partial^2\mathrm{u}}{\partial x\partial\phi}+\frac{1}{a^2}\frac{\partial\mathrm{w}}{\partial\phi}=0,\\
&\qquad\frac{\kappa^2}{a^2}\mathrm{w}-\frac{1}{a^2}\mathrm{w}-\frac{1}{a}\left(\sigma\frac{\partial\mathrm{u}}{\partial x}+\frac{1}{a}\frac{\partial\mathrm{v}}{\partial\phi}\right)=0
\end{aligned}\right\}\text{...(18).}$$

Eliminating w from the first two equations by means of the third, we find

$$\left.\begin{aligned}
&\frac{\partial^2\mathrm{u}}{\partial x^2}+\frac{\kappa^2}{a^2}\mathrm{u}+\tfrac{1}{2}(1-\sigma)\frac{1}{a^2}\frac{\partial^2\mathrm{u}}{\partial\phi^2}+\tfrac{1}{2}(1+\sigma)\frac{1}{a}\frac{\partial^2\mathrm{v}}{\partial x\partial\phi}\\
&\qquad\qquad+\frac{\sigma}{\kappa^2-1}\left[\sigma\frac{\partial^2\mathrm{u}}{\partial x^2}+\frac{1}{a}\frac{\partial^2\mathrm{v}}{\partial x\partial\phi}\right]=0,\\
&\frac{1}{a^2}\frac{\partial^2\mathrm{v}}{\partial\phi^2}+\frac{\kappa^2}{a^2}\mathrm{v}+\tfrac{1}{2}(1-\sigma)\frac{\partial^2\mathrm{v}}{\partial x^2}+\tfrac{1}{2}(1+\sigma)\frac{1}{a}\frac{\partial^2\mathrm{u}}{\partial x\partial\phi}\\
&\qquad\qquad+\frac{1}{\kappa^2-1}\left[\frac{\sigma}{a}\frac{\partial^2\mathrm{u}}{\partial x\partial\phi}+\frac{1}{a^2}\frac{\partial^2\mathrm{v}}{\partial\phi^2}\right]=0
\end{aligned}\right\}\text{...(19).}$$

In the case of an infinite cylinder[1] we have to take as the type of solution

$$\left.\begin{aligned}
\mathrm{u}&=A\cos(mx+x_0)\cos(n\phi+\phi_0)\,e^{\iota pt},\\
\mathrm{v}&=B\sin(mx+x_0)\sin(n\phi+\phi_0)\,e^{\iota pt}
\end{aligned}\right\}\text{.........(20).}$$

If we substitute these values for u and v, and eliminate A and B, we obtain an equation for κ^2, which is

$$\begin{aligned}
&\left[(\kappa^2-1)\left\{\frac{\kappa^2}{a^2}-m^2-\tfrac{1}{2}(1-\sigma)\frac{n^2}{a^2}\right\}-\sigma^2m^2\right]\\
&\qquad\qquad\left[(\kappa^2-1)\left\{\frac{\kappa^2-n^2}{a^2}-\tfrac{1}{2}(1-\sigma)\,m^2\right\}-\frac{n^2}{a^2}\right]\\
&\qquad=\left[\tfrac{1}{2}(\kappa^2-1)(1+\sigma)+\sigma\right]^2\frac{m^2n^2}{a^2}\text{...............(21).}
\end{aligned}$$

This is the general frequency-equation.

Among particular cases[2] we may notice the following:—

(1°) Radial vibrations.

Taking $\mathrm{u}=0$, $\mathrm{v}=0$, and w independent of x and ϕ, we find the frequency given by the equation

$$p^2=\frac{2\mu}{(1-\sigma)\,a^2\rho}.$$

[1] See Lord Rayleigh's 'Note on the Free Vibrations of an infinitely long Cylindrical Shell'. *Proc. R. S.* XLV. 1889.

[2] The results only are stated and the analysis is left to the reader. Except for the radial vibrations the results are due to Lord Rayleigh.

(2°) Vibrations in which the displacements are independent of x.

These fall into two sub-cases as follows:

(a) We may take v and w to vanish while

$$u = A \cos(n\phi + \phi_0) e^{\iota pt},$$

then the frequency is given by the equation

$$p^2 = \frac{n^2}{a^2}\frac{\mu}{\rho}.$$

In this motion every generator moves parallel to itself through a distance which is a simple harmonic function of the arc of the circular section measured from a fixed generator.

(b) We may take $u = 0$, while v and w are independent of x. The motion is in two dimensions, and the equation for κ becomes

$$\kappa^2[\kappa^2 - (1 + n^2)] = 0.$$

The root $\kappa = 0$ indicates indefinitely slow motion, and we shall find that there is a corresponding flexural mode; the other root gives a frequency determined by the equation

$$p^2 = \frac{2}{1-\sigma}\frac{1+n^2}{a^2}\frac{\mu}{\rho}.$$

(3°) Vibrations in which the displacements are independent of ϕ.

These fall into two sub-cases as follows:—

(a) We may take u and w to vanish, while

$$v = B \sin(mx + x_0) e^{\iota pt},$$

then the frequency is given by the equation

$$p^2 = m^2 \frac{\mu}{\rho}.$$

The motion is purely tangential, each circular section moving along itself. These vibrations correspond to the torsional vibrations of a thin rod (art. 264).

(b) We may take $v = 0$, while u and w are independent of ϕ, then the values of κ are given by the equation

$$(\kappa^2 - 1)(\kappa^2 - m^2a^2) - \sigma^2 m^2 a^2 = 0.$$

When ma is small or the wave-length large compared with the radius the two values of κ correspond respectively to vibrations which are almost purely radial with a frequency nearly equal to

$$\frac{1}{2\pi a}\sqrt{\frac{\mu}{2\rho(1-\sigma)}},$$

and vibrations in which the displacement is almost wholly parallel to the axis with a frequency given by

$$p^2 = \frac{E}{\rho} m^2,$$

where E is the Young's modulus of the material. The latter correspond to the extensional vibrations of a thin rod (art. 263).

The extensional vibrations of a cylinder of finite length are considered in my paper (*Phil. Trans. A.* 1888). Taking the case where v and w vanish at the end $x=0$, while the end $x=l$ is free, the symmetrical vibrations can be completely determined. For this case we have

$$\text{u} = A\cos mx\, e^{\iota pt}, \qquad \text{v} = B\sin mx\, e^{\iota pt},$$

$$\text{w} = -\frac{A\sigma ma}{\kappa^2-1}\sin mx\, e^{\iota pt}.$$

The boundary-conditions at $x=l$ are $\epsilon_1+\sigma\epsilon_2=0$, $\varpi=0$, or

$$-A\left(1+\frac{\sigma^2}{\kappa^2-1}\right)\sin ml = 0, \qquad B\cos ml = 0.$$

To satisfy these we must take either

$$ml = n\pi, \qquad B = 0,$$

where n is an integer, and then κ is given by the equation

$$(\kappa^2-1)\left(\kappa^2-\frac{n^2\pi^2a^2}{l^2}\right)-\sigma^2\frac{n^2\pi^2a^2}{l^2}=0;$$

or else we must take

$$ml = \tfrac{1}{2}(2n+1)\pi, \qquad A = 0,$$

and then κ is given by the equation

$$\kappa^2 = \tfrac{1}{8}(1-\sigma)(2n+1)^2\pi^2a^2/l^2.$$

These correspond to the modes of vibration of an infinite cylinder described in (3°), (*b*), and (3°), (*a*) respectively.

352. Non-extensional Vibrations[1].

We proceed to consider the modes of deformation of the cylindrical shell in which no line on the middle-surface is altered in length. These are characterised by the equations of inextensibility

$$\frac{\partial \mathrm{u}}{\partial x}=0, \qquad \frac{\partial \mathrm{v}}{\partial \phi}+\mathrm{w}=0, \qquad \frac{1}{a}\frac{\partial \mathrm{u}}{\partial \phi}+\frac{\partial \mathrm{v}}{\partial x}=0 \text{.........(22)}.$$

The cylinder being complete u, v, w must be periodic in regard to ϕ with period 2π, and therefore must consist of sums of terms which are simple harmonic functions of integral multiples of ϕ.

Further, by differentiating the third of equations (22) with respect to x, we find

$$\frac{\partial^2 \mathrm{v}}{\partial x^2}=0,$$

and hence we have to take as the most general forms for u, v, w

$$\left.\begin{aligned} \mathrm{u} &= -\Sigma \frac{a}{n} B_n \sin(n\phi+\phi_0), \\ \mathrm{v} &= \ \ \Sigma\ (A_n+B_n x)\cos(n\phi+\phi_0), \\ \mathrm{w} &= \ \ \Sigma n\,(A_n+B_n x)\sin(n\phi+\phi_0) \end{aligned}\right\} \text{............ (23)},$$

where the summation refers to integral values for n.

With these values we find from (13)

$$\kappa_1=0, \qquad a^2\kappa_2=-\Sigma\,(n^3-n)\,(A_n+B_n x)\sin(n\phi+\phi_0),$$
$$a\tau=\Sigma\,(n^2-1)\,B_n\cos(n\phi+\phi_0) \text{............(24)}.$$

The modes of non-extensional deformation fall into two classes. In the first class, for which the coefficients B_n are zero, the motion is in two dimensions, and is analogous to the non-extensional deformation of a circular ring considered in art. 303; in the second class, for which the coefficients A_n are zero, the displacements in the plane of any circular section are proportional to the distance from a particular circular section, so that these cannot occur in an infinite cylinder.

For a cylinder of length $2l$ bounded by two circles $x=l$ and $x=-l$ the kinetic energy is

$$2\pi\rho h l a \Sigma\left[\dot{A}_n^2(1+n^2)+\dot{B}_n^2\left\{(1+n^2)\frac{l^2}{3}+\frac{a^2}{n^2}\right\}\right] \text{......(25)},$$

where the dot denotes differentiation with respect to the time.

[1] See Lord Rayleigh 'On the Bending and Vibrations of thin Elastic Shells especially of Cylindrical Form'. *Proc. R. S.* XLV. 1889.

The potential energy for the same cylinder is

$$\pi Cla\Sigma\left[\frac{A_n^2}{a^4}n^2(n^2-1)^2+\frac{B_n^2}{a^4}\left\{n^2(n^2-1)^2\frac{l^2}{3}+2(1-\sigma)(n^2-1)^2a^2\right\}\right] \quad \text{.......................(26).}$$

Since both these expressions are sums of squares A_n and B_n may be treated as normal coordinates of the system.

The frequency $p/2\pi$ for the two-dimensional vibration, in which A_n only is different from zero is given by the equation

$$p^2=\tfrac{1}{2}\frac{C}{\rho ha^4}\frac{n^2(n^2-1)^2}{n^2+1}=\frac{4\mu(\lambda+\mu)}{3(\lambda+2\mu)}\frac{h^2}{\rho a^4}\frac{n^2(n^2-1)^2}{n^2+1} \quad \text{...(27),}$$

and the frequency $p'/2\pi$ for a vibration in which B_n only is different from zero is given by the equation

$$p'^2=\tfrac{1}{2}\frac{C}{\rho ha^4}\frac{n^2(n^2-1)^2}{n^2+1}\frac{1+(1-\sigma)\dfrac{6a^2}{n^2l^2}}{1+\dfrac{3a^2}{n^2(n^2+1)l^2}} \quad \text{..........(28).}$$

Lord Rayleigh remarks that when the length of the cylinder is at all great compared with its diameter p'^2 differs very little from p^2. If, however, the length be too great the motion in B_n tends to become finite, and it will not be legitimate in the equations of inextensibility to neglect squares of the displacements.

353. Discussion of the two-dimensional vibrations.

We have now to consider to what extent the equations of small motion are satisfied by the non-extensional solutions.

Take first the two-dimensional deformation expressed by the equations

$$\mathrm{u}=0, \qquad \mathrm{v}=A\cos n\phi\, e^{ipt}, \qquad \mathrm{w}=nA\sin n\phi\, e^{ipt} \quad \text{...(29),}$$

for which ϵ_1, ϵ_2, ϖ all vanish, and

$$\kappa_1=0, \qquad a^2\kappa_2=-(n^3-n)A\sin n\phi\, e^{ipt}, \qquad \tau=0.$$

With these expressions for the displacements and changes of curvature, the stress-resultants and stress-couples are given by the equations

$$\left.\begin{aligned}
P_1 &= -\tfrac{1}{2}C\frac{\sigma^2}{1-\sigma}\frac{n^3-n}{a^3}A\sin n\phi e^{\iota pt} + \tfrac{1}{2}C\frac{\sigma R_0}{\mu},\\
P_2 &= -\tfrac{1}{2}C\frac{2-\sigma}{1-\sigma}\frac{n^3-n}{a^3}A\sin n\phi e^{\iota pt} + \tfrac{1}{2}C\frac{\sigma R_0}{\mu},\\
U_1 &= U_2 = 0, \qquad H = 0,\\
G_1 &= \frac{C\sigma}{a^2}(n^3-n)A\sin n\phi\, e^{\iota pt},\quad G_2 = -\frac{C}{a^2}(n^3-n)A\sin n\phi\, e^{\iota pt}
\end{aligned}\right\}\ldots(30).$$

The equations of small motion become

$$\left.\begin{aligned}
&\frac{\partial P_1}{\partial x} = 0,\\
&\frac{1}{a}\frac{\partial P_2}{\partial\phi} + \frac{C}{a^4}n^2(n^2-1)A\cos n\phi\, e^{\iota pt} = -2\rho hp^2 A\cos n\phi\, e^{\iota pt},\\
&-\frac{P_2}{a} - \frac{C}{a^4}n^3(n^2-1)A\sin n\phi\, e^{\iota pt} = -2\rho hnp^2 A\sin n\phi\, e^{\iota pt}
\end{aligned}\right\}\ldots(31).$$

Differentiating the third with respect to ϕ, and adding to the second, we eliminate P_2 and obtain the equation

$$\left[2\rho h(n^2+1)p^2 - \frac{C}{a^4}n^2(n^2-1)^2\right]A\cos n\phi\, e^{\iota pt} = 0.$$

We observe that in this case the differential equations of vibration are satisfied by the non-extensional solution, and the substitution of this solution in the said equations leads to the correct expression for the frequency. With some set of boundary-conditions the equations (29) give us a system of normal functions.

Before proceeding to the consideration of the boundary-conditions we may find the stresses P_1, P_2, and the unknown R_0.

We have[1] on substituting for p^2 in the third of equations (31),

$$P_2 = -\frac{2n^3(n^2-1)}{n^2+1}\frac{C}{a^3}A\sin n\phi\, e^{\iota pt},$$

[1] This is the particular case referred to in art. 339 in which P_2 can be determined and shewn to be in disagreement with Mr Basset's result (*Phil. Trans.* A. 1890). On his p. 456 are three equations which are really the same as the second and third of (31) above, and the equation of (30) that gives G_1, and they would give rise to the same expression for P_2 (Mr Basset's T_2) as that obtained here. Yet the expression he obtains for his T_2 in the second of his equations (44) on p. 450 is not satisfied by this value of his T_2.

and then from the first and second of (30),

$$\frac{\sigma R_0}{\mu}=\frac{n(n^2-1)}{a^3}\left(\frac{2-\sigma}{1-\sigma}-\frac{4n^2}{n^2+1}\right)A\sin n\phi\, e^{\iota pt},$$

$$P_1=C\frac{n(n^2-1)}{a^3}\left(1+\tfrac{1}{2}\sigma-\frac{2n^2}{n^2+1}\right)A\sin n\phi\, e^{\iota pt}.$$

The result is interesting as shewing that there is a tension P_1 across the circular sections of the cylinder as well as a tension P_2 across the generators.

As regards the boundary-conditions it is clear that if any circular section be a free edge the conditions there cannot be satisfied. Among these conditions are $P_1=0$, $G_1=0$, and neither of these quantities vanishes at all points of any circle $x=\text{const.}$

If a certain stress-system be always applied to the edge the motion will be possible, but otherwise it can only take place when the cylinder is of infinite length. There is reason, however, for thinking that, by means of a correction to the displacements which is unimportant except quite close to the edges, forms could be obtained by which the boundary-conditions could be satisfied without any sensible change being produced in the period or in the general character of the motion. (See art. 349.)

354. Discussion of the three-dimensional vibrations.

Let us consider next the deformation expressed by the equations

$$\mathrm{u}_0=-\frac{a}{n}B\sin n\phi\, e^{\iota pt},\quad \mathrm{v}_0=Bx\cos n\phi\, e^{\iota pt},\quad \mathrm{w}_0=nBx\sin n\phi\, e^{\iota pt},$$

and suppose that u, v, w are given by

$$\mathrm{u}=\mathrm{u}_0+h^2\mathrm{u}',\quad \mathrm{v}=\mathrm{v}_0+h^2\mathrm{v}',\quad \mathrm{w}=\mathrm{w}_0+h^2\mathrm{w}',$$

where u′, v′, w′ are at present undetermined functions of x and ϕ. Then ϵ_1, ϵ_2, ϖ depend on u′, v′, w′ and not on u_0, v_0, w_0, while κ_1, κ_2, τ are given by the equations

$$\left.\begin{aligned}\kappa_1&=h^2\frac{\partial^2\mathrm{w}'}{\partial x^2},\\ \kappa_2&=-\frac{(n^3-n)}{a^2}Bx\sin n\phi\, e^{\iota pt}+\frac{h^2}{a^2}\left(\frac{\partial^2\mathrm{w}'}{\partial\phi^2}-\frac{\partial\mathrm{v}'}{\partial\phi}\right),\\ \tau&=\frac{n^2-1}{a}B\cos n\phi\, e^{\iota pt}+\frac{h^2}{a}\left(\frac{\partial^2\mathrm{w}'}{\partial x\partial\phi}-\frac{\partial\mathrm{v}'}{\partial x}\right)\end{aligned}\right\}.$$

We may omit the terms in h^2 from the expressions for the stress-couples, and thus find

$$\left.\begin{aligned} G_1 &= C\sigma\,\frac{n^3-n}{a^2}\,Bx\sin n\phi\, e^{\iota pt},\\ G_2 &= -C\,\frac{n^3-n}{a^2}\,Bx\sin n\phi\, e^{\iota pt},\\ H &= C(1-\sigma)\frac{n^2-1}{a}\,B\cos n\phi\, e^{\iota pt} \end{aligned}\right\}\ldots\ldots\ldots\ldots (32).$$

Now in the equations of small motion (2) the quantities P_1, P_2, U_1, U_2 contain unknowns ϵ_1, ϵ_2, R_0, ϖ, so that we may first write down equations for determining P_1, ... and afterwards seek the values of u′, v′, w′. In these equations the terms arising from kinetic reactions may be simplified by omitting u′, v′, w′, since p^2 is proportional to h^2.

The equations are

$$\left.\begin{aligned} &\frac{\partial P_1}{\partial x}+\frac{1}{a}\frac{\partial U_2}{\partial \phi}=2\rho hp^2\frac{a}{n}B\sin n\phi\, e^{\iota pt},\\ &\frac{\partial U_1}{\partial x}+\frac{1}{a}\frac{\partial P_2}{\partial \phi}+\frac{C}{a^4}n^2(n^2-1)Bx\cos n\phi\, e^{\iota pt}=-2\rho hp^2Bx\cos n\phi\, e^{\iota pt},\\ &-\frac{P_2}{a}-\frac{C}{a^4}n^3(n^2-1)Bx\sin n\phi\, e^{\iota pt}=-2\rho hp^2nBx\sin n\phi\, e^{\iota pt} \end{aligned}\right\} (33),$$

in which P_1, P_2, U_1, U_2 contain terms arising from the terms h^2u′, h^2v′, h^2w′ in the displacements.

The third of equations (33) gives P_2. Eliminating P_2 between the second and third we find

$$\frac{\partial U_1}{\partial x}=\left[\frac{C}{a^4}n^2(n^2-1)^2-2\rho hp^2(n^2+1)\right]Bx\cos n\phi\, e^{\iota pt}.$$

This gives

$$U_1=\tfrac{1}{2}\left[\frac{C}{a^4}n^2(n^2-1)^2-2\rho hp^2(n^2+1)\right]Bx^2\cos n\phi\, e^{\iota pt}+\tfrac{1}{2}U_0\cos n\phi\, e^{\iota pt},$$

where U_0 is independent of x, and the ϕ and t factors are inserted with a view to satisfying conditions at $x=\pm l$.

If $H_0\cos n\phi\, e^{\iota pt}$ be the value of H at either limit we have as one of the boundary-conditions

$$U_1-\frac{H_0}{a}\cos n\phi\, e^{\iota pt}=0 \quad\text{when}\quad x=\pm l,$$

so that

$$U_0 = \frac{2H_0}{a} - \frac{Cl^2}{a^4}\,[n^2(n^2-1)^2 - 2\rho h p^2 (n^2+1)].$$

Now, by (15) of art. 350,

$$U_2 - U_1 = C(1-\sigma)\,\tau/a,$$

so that

$$U_2 = \Big\{C(1-\sigma)\frac{n^2-1}{a^2}B + \frac{H_0}{a} - \tfrac{1}{2}\left[\frac{C}{a^4}n^2(n^2-1)^2 - 2\rho h p^2(n^2+1)\right](l^2-x^2)B\Big\}\cos n\phi\, e^{\iota pt}.$$

Hence the first of equations (33) becomes

$$\frac{\partial P_1}{\partial x} = \Big[2\rho h p^2\left\{\frac{a}{n} + \tfrac{1}{2}n(n^2+1)\frac{l^2-x^2}{a}\right\}B - \frac{C}{a^3}n(n^2-1)\left\{\tfrac{1}{2}n^2(n^2-1)\frac{l^2-x^2}{a^2} - (1-\sigma)\right\}B + n\frac{H_0}{a^2}\Big]\sin n\phi\, e^{\iota pt}.$$

Hence, on integrating, we have

$$P_1 = x\Big[n\frac{H_0}{a^2} + 2\rho h p^2\left\{\frac{a}{n} + \tfrac{1}{2}\frac{n(n^2+1)}{a}(l^2 - \tfrac{1}{3}x^2)\right\}B - \frac{C}{a^3}n(n^2-1)\left\{\tfrac{1}{2}\frac{n^2(n^2-1)}{a^2}(l^2-\tfrac{1}{3}x^2) - (1-\sigma)\right\}B\Big]\sin n\phi\, e^{\iota pt} + P_0 \sin n\phi\, e^{\iota pt},$$

where P_0 is independent of x.

To make P_1 vanish when $x = \pm l$ we must take $P_0 = 0$, and

$$H_0 = na^3B\left[\frac{C}{a^4}\left\{\frac{n^2(n^2-1)^2l^2}{3a^2} - (1-\sigma)(n^2-1)\right\} - 2\rho h p^2\left\{\frac{1}{n^2} + \frac{(n^2+1)l^2}{3a^2}\right\}\right].$$

Now by (28) we have

$$p^2 = \tfrac{1}{2}\frac{C}{\rho h a^4}\left\{\frac{n^2(n^2-1)^2l^2}{3a^2} + 2(1-\sigma)(n^2-1)^2\right\} \Big/ \left\{\frac{1}{n^2} + \frac{(n^2+1)l^2}{3a^2}\right\}.$$

Hence $$H_0 = -na^3B\,\frac{C(1-\sigma)}{a^4}(n^2-1)(2n^2-1),$$

and the value of H at a free edge is

$$-C(1-\sigma)\,n(2n^2-1)\frac{B}{a}(n^2-1)\cos n\phi\, e^{\iota pt}.$$

This does not agree with the value of H given in (32), and we must conclude that near a free edge there exist subsidiary displacements which give rise to terms in τ of the same order as

those given by the non-extensional solution u_0, v_0, w_0; these displacements are required to satisfy the boundary-conditions.

Again, we have, by (15) of art. 350,

$$U_2 = \frac{3C}{2h^2}(1-\sigma)\varpi + \tfrac{1}{2}C(1-\sigma)\frac{n^2-1}{a^2}B\cos n\phi\, e^{ipt},$$

and we have found a different expression for U_2, which shews that ϖ is of the order h^2B/a^2. In like manner from the values of P_1 and P_2 we could shew that ϵ_1 and ϵ_2 are of the order h^2B/a^2. We conclude that there exist subsidiary displacements which are of the order h^2u_0/a^2 without which it is impossible to satisfy the differential equations.

355. Effect of free edges.

We shall now, following Prof. Lamb[1], investigate a statical problem which illustrates the effect of free edges. We shall suppose a part of a circular cylinder of radius a bounded by two generators ($\phi = \pm\phi_0$) and two circular sections ($x = \pm l$) to be held bent into a surface of revolution by forces applied along the two generators while the circular edges are free. The type of displacement will be expressed by assuming that

$$v = c\phi,$$

while u and w are independent of ϕ.

We have then

$$\epsilon_1 = \frac{\partial u}{\partial x}, \qquad \epsilon_2 = \frac{w+c}{a}, \qquad \varpi = 0,$$

$$\kappa_1 = \frac{\partial^2 w}{\partial x^2}, \qquad \kappa_2 = -\frac{c}{a^2}, \qquad \tau = 0.$$

Each of these quantities is independent of ϕ.

Of the stress-resultants and stress-couples U_1, U_2, and H vanish, while P_2, G_1, and G_2 are independent of ϕ, also no forces are applied to the shell except at the straight edges. Hence the second of equations (14) is identically satisfied and the other two become

$$\frac{\partial P_1}{\partial x} = 0,$$

$$\frac{\partial^2 G_1}{\partial x^2} - \frac{P_2}{a} = 0,$$

[1] *Proc. Lond. Math. Soc.* XXI. 1890.

The boundary-conditions which hold at the circular edges are that, when $x = \pm l$,

$$P_1 = 0, \qquad G_1 = 0, \qquad T_1 - \frac{1}{a}\frac{\partial H}{\partial \phi} = 0, \qquad U_1 - H/a = 0,$$

of which the last is satisfied identically, and the third is

$$\frac{\partial G_1}{\partial x} = 0,$$

We propose to solve these on the assumption that ϵ_1, ϵ_2 are of the order $h\kappa_1$, and we shall find a justification of this assumption *a posteriori*.

With this assumption we have

$$P_1 = \frac{3C}{h^2}(\epsilon_1 + \sigma\epsilon_2), \qquad P_2 = \frac{3C}{h^2}(\epsilon_2 + \sigma\epsilon_1), \qquad G_1 = -C(\kappa_1 + \sigma\kappa_2).$$

The equation $\dfrac{\partial P_1}{\partial x} = 0$, with the conditions that $P_1 = 0$ when $x = \pm l$, gives us $P_1 = 0$ everywhere, and we therefore have

$$\epsilon_1 = -\sigma\epsilon_2.$$

The equation $\dfrac{\partial^2 G_1}{\partial x^2} - \dfrac{P_2}{a} = 0$ then gives us

$$\frac{\partial^4 \mathrm{w}}{\partial x^4} + \frac{3}{h^2}(1-\sigma^2)\frac{\mathrm{w} + c}{a^2} = 0.$$

The solution of this is in terms of exponentials of $\pm(1 \pm \iota)\, qx/a$, where

$$q^2 = \frac{a}{h}\frac{\sqrt{3}}{2}\sqrt{(1-\sigma^2)};$$

and, selecting the terms which are even functions of x, we have

$$\mathrm{w} + c = D_1 \cosh\frac{qx}{a}\cos\frac{qx}{a} + D_2 \sinh\frac{qx}{a}\sin\frac{qx}{a},$$

where D_1 and D_2 are arbitrary constants.

The boundary-conditions, $G_1 = 0$ and $\partial G_1/\partial x = 0$ when $x = \pm l$, are

$$\frac{\partial^2 \mathrm{w}}{\partial x^2} - \frac{\sigma c}{a^2} = 0, \qquad \frac{\partial^3 \mathrm{w}}{\partial x^3} = 0;$$

from which we find

$$D_1 = \frac{\sigma c}{q^2}\,\frac{\sinh ql/a \cos ql/a - \cosh ql/a \sin ql/a}{\sinh 2ql/a + \sin 2ql/a},$$

$$D_2 = \frac{\sigma c}{q^2}\,\frac{\sinh ql/a \cos ql/a + \cosh ql/a \sin ql/a}{\sinh 2ql/a + \sin 2ql/a}.$$

This constitutes the complete solution of the problem, and the *a posteriori* justification is to be found in the fact that when x is nearly equal to $\pm l$, ϵ_2 or $(\mathrm{w}+c)/a$ is actually of the order $h\kappa_2$ or hc/a^2.

Now since q is very great the most important terms in w are given by

$$\mathrm{w}=-c+\frac{\sigma c}{q^2}e^{-q(l-x)/a}\left[\left(\cos\frac{ql}{a}-\sin\frac{ql}{a}\right)\cos\frac{qx}{a}\right.$$
$$\left.+\left(\cos\frac{ql}{a}+\sin\frac{ql}{a}\right)\sin\frac{qx}{a}\right],$$

or
$$\mathrm{w}=-c+\sqrt{2}\,\frac{\sigma c}{q^2}e^{-q(l-x)/a}\cos\left\{q(l-x)/a+\frac{\pi}{4}\right\}.$$

The changes of curvature near the free edges are of the order c/a^2, and they are of the same order all over the surface.

The extensions near the free edges are of the order hc/a^2, and they rapidly diminish in value as we pass inwards from the edges.

It may be shewn that the total potential energy arising from the extensions near the edges is of the order $\sqrt{(h/a)}$ as compared with that arising from the changes of curvature.

It may also be shewn that the forces required to hold the shell in the form investigated reduce to uniform flexural couples along the bounding generators and forces whose effects are negligible at a very little distance from those generators. The proofs of the last two statements are left to the reader.

Prof. Lamb[1] has given an interesting application of his analysis to the case of a plane plate bent into a cylinder of finite curvature. When the length and breadth of the plate are of the same order of magnitude, the plate can be held exactly in the cylindrical form by the application of couples at the edges. Along the straight edges the couple is the product of the cylindrical rigidity C and the curvature, along the circular edges the couple resisting the tendency to anticlastic curvature (art. 314) is the product of the Poisson's ratio σ, the cylindrical rigidity C, and the curvature. If the circular edges be free the departure from the cylindrical form is everywhere very slight, but near the free edges there is an extensional strain of the same order of magnitude as the flexural strain. When the plate becomes a narrow strip whose breadth is

[1] 'On the Flexure of a Flat Elastic Spring'. *Proc. Phil. Soc.* Manchester, 1890.

comparable with a mean proportional between the radius of curvature and the thickness, the extensional strain is of importance at all points of the strip. When the breadth of the strip is still further reduced it bends like a thin rod.

356. Further examples of cylindrical shells.

We conclude our discussion of cylindrical shells with the following examples[1]:

(1°) An elongated cylindrical strip bounded by two generators cannot be bent in the plane containing the axis and the middle-generator.

(2°) The form of the potential energy (26) of art. 353 suffices for the solution of many statical problems. Thus if a cylinder of length $2l$ be compressed along the diameter through $x=x_0$, $\phi=0$ by equal normal forces F the normal displacement w is given by

$$\mathrm{w} = \sum_1^\infty 2n\,(A_{2n} + xB_{2n}) \cos 2n\phi,$$

where $$A_{2n} = -\frac{a^3}{2\pi l}\frac{F}{C}\frac{1}{n\,(4n^2-1)^2},$$

$$B_{2n} = -\frac{ax_0}{2\pi l}\frac{F}{C}\frac{1}{n\,(4n^2-1)^2\,\{l^2/3a^2+(1-\sigma)/2n\}}.$$

(3°) When a heavy cylinder of infinite length whose section is a semicircle is supported with its axis horizontal by uniform tensions applied to its straight edges, it can be shewn that the change of curvature at an angular distance ϕ from the lowest point is

$$\frac{2g\rho a^2 h}{C}(\phi \sin\phi + \cos\phi - \tfrac{1}{2}\pi),$$

and that the strain is non-extensional.

[1] The results only are stated and the verification is left to the reader. The first two examples are taken from Lord Rayleigh's paper 'On the Bending and Vibrations of Thin Elastic Shells especially of Cylindrical Form'. *Proc. R. S.* XLV. 1888. The third is from Mr Basset's paper 'On the Extension and Flexure of Cylindrical and Spherical Thin Elastic Shells'. *Phil. Trans. R. S.* (A), 1890. Mr Basset makes the strain only approximately non-extensional, the departure from a condition in which no line on the middle-surface is altered in length depending on the fact that the centre of gravity of a small element does not lie on the middle-surface. It is however unnecessary to take this into account.

SPHERICAL SHELLS.

357. We shall now consider the case of a shell whose middle-surface when unstrained is spherical. We shall first investigate expressions for the displacements when no line on the middle-surface is altered in length, and deduce an expression for the potential energy of bending. This expression will be applied to find the frequencies of the non-extensional vibrations, and to some statical problems. We shall then consider the subsidiary displacements which must be superposed upon the non-extensional displacements in order to satisfy the equations of vibration. After that we shall proceed with the extensional vibrations and with some statical problems in which extension plays an important part.

358. Conditions of inextensibility[1].

Referring to the notation of the general theory we shall take θ and ϕ for the colatitude and longitude of a point of the sphere, and we shall take

$$\alpha = \theta, \quad \beta = \phi \qquad \text{.............................(34)},$$

then, if the normal be supposed drawn outwards we have,

$$A = a, \quad B = a\sin\theta, \quad \rho_1 = \rho_2 = -a \qquad \text{............(35)},$$

a being the radius of the unstrained spherical surface.

Let u, v, w be the displacements of a point on the middle-surface along the tangent to the meridian, the tangent to the parallel, and the normal to the sphere through the point, then the condition that no line on the middle-surface is altered in length is expressed by the equations

$$\left.\begin{aligned} \frac{\partial \mathrm{u}}{\partial \theta} + \mathrm{w} = 0,\\ \frac{1}{\sin\theta}\frac{\partial \mathrm{v}}{\partial \phi} + \mathrm{u}\cot\theta + \mathrm{w} = 0,\\ \frac{1}{\sin\theta}\frac{\partial \mathrm{u}}{\partial \phi} + \frac{\partial \mathrm{v}}{\partial \theta} - \mathrm{v}\cot\theta = 0 \end{aligned}\right\} \qquad \text{................(36)}.$$

These are derived from (18) of art. 330 by substituting for α, β, A, B, ρ_1, ρ_2 their values as given in (34) and (35).

Eliminating w we arrive at two equations which can be put in the form:

[1] See Lord Rayleigh in *Proc. Lond. Math. Soc.* XIII. 1882.

$$\left.\begin{aligned}\frac{\partial}{\partial\phi}\left(\frac{\mathrm{u}}{\sin\theta}\right)+\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\mathrm{v}}{\sin\theta}\right)=0,\\ \frac{\partial}{\partial\phi}\left(\frac{\mathrm{v}}{\sin\theta}\right)-\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\mathrm{u}}{\sin\theta}\right)=0\end{aligned}\right\}\text{............(37)},$$

and these shew that $\mathrm{u}\operatorname{cosec}\theta$ and $\mathrm{v}\operatorname{cosec}\theta$ are conjugate functions of $\log(\tan\frac{1}{2}\theta)$ and ϕ, or, in other words, that $(\mathrm{u}+\iota\mathrm{v})\operatorname{cosec}\theta$ is a function of $\log(\tan\frac{1}{2}\theta)+\iota\phi$. We shall consider especially the case where the shell is bounded by either one or two small circles $\theta=$const., but the application to other cases might be made by means of a transformation by conjugate functions of $\log(\tan\frac{1}{2}\theta)$ and ϕ.

For the case supposed u and v as functions of ϕ must be capable of expansion in sines or cosines of integral multiples of ϕ, and we have the general forms[1]:

$$\left.\begin{aligned}\mathrm{u}&=\sin\theta\Sigma\left[A_n\tan^n\tfrac{1}{2}\theta+B_n\cot^n\tfrac{1}{2}\theta\right]^{\cos}_{-\sin}n\phi,\\ \mathrm{v}&=\sin\theta\Sigma\left[A_n\tan^n\tfrac{1}{2}\theta+B_n\cot^n\tfrac{1}{2}\theta\right]^{\sin}_{\cos}n\phi,\\ \mathrm{w}&=-\Sigma\left[(\cos\theta+n)\left(A_n\tan^n\tfrac{1}{2}\theta+B_n\cot^n\tfrac{1}{2}\theta\right)\right]^{\cos}_{-\sin}n\phi\end{aligned}\right\}\text{...(38)}.$$

The form of the results shews that in order that u, v, w may be finite both the poles $\theta=0$ and $\theta=\pi$ cannot be included. This is in accordance with the well-known result that a closed surface cannot be bent without stretching.

359. Potential Energy.

We can next calculate the potential energy of bending. Referring to equations (25) of art. 332 we see that in the case of the sphere the changes of curvature are given by

$$\left.\begin{aligned}a^2\kappa_2&=\frac{1}{\sin^2\theta}\frac{\partial^2\mathrm{w}}{\partial\phi^2}+\cot\theta\frac{\partial\mathrm{w}}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial\mathrm{v}}{\partial\phi}-\mathrm{u}\cot\theta,\\ a^2\kappa_1&=\frac{\partial^2\mathrm{w}}{\partial\theta^2}-\frac{\partial\mathrm{u}}{\partial\theta},\\ a^2\tau&=\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\mathrm{w}}{\partial\phi}\right)\end{aligned}\right\}\text{...(39)}.$$

[1] Written out in full

$$\mathrm{u}=\sin\theta\Sigma\left[(A_n\tan^n\tfrac{1}{2}\theta+B_n\cot^n\tfrac{1}{2}\theta)\cos n\phi-(A_n'\tan^n\tfrac{1}{2}\theta+B_n'\cot^n\tfrac{1}{2}\theta)\sin n\phi\right],$$

and similarly for v and w.

By means of equations (36) these can be reduced to

$$\left.\begin{aligned} a^2\kappa_2 &= -\frac{1}{\sin^2\theta}\frac{\partial^3 \mathrm{u}}{\partial\theta\partial\phi^2} - \cot\theta\frac{\partial^2 \mathrm{u}}{\partial\theta^2} - \frac{\partial \mathrm{u}}{\partial\theta}, \\ a^2\kappa_1 &= -\frac{\partial^3 \mathrm{u}}{\partial\theta^3} - \frac{\partial \mathrm{u}}{\partial\theta}, \\ a^2\tau &= -\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial^2 \mathrm{u}}{\partial\theta\partial\phi}\right) \end{aligned}\right\}\quad\text{.........(40)},$$

Now on eliminating v from (37) we have

$$\sin^2\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\mathrm{u}}{\sin\theta}\right)\right] + \frac{\partial^2 \mathrm{u}}{\partial\phi^2} = 0 \text{...........(41)},$$

and, by differentiating this with respect to θ and dividing by $\sin^2\theta$, we find an equation which is

$$\kappa_1 + \kappa_2 = 0.$$

We now have, on differentiating,

$$\left.\begin{aligned} a^2\kappa_2 = -a^2\kappa_1 &= \operatorname{cosec}^2\theta\,\Sigma\,(n^3 - n)\left[A_n \tan^n \tfrac{1}{2}\theta + B_n \cot^n \tfrac{1}{2}\theta\right] {}^{\cos}_{-\sin}\, n\phi, \\ a^2\tau &= \operatorname{cosec}^2\theta\,\Sigma\,(n^3 - n)\left[A_n \tan^n \tfrac{1}{2}\theta + B_n \cot^n \tfrac{1}{2}\theta\right] {}^{\sin}_{\cos}\, n\phi \end{aligned}\right\}$$

$$\text{..................(42)}.$$

The potential energy per unit area is

$$\begin{aligned} &\tfrac{1}{2}C\left[(\kappa_1+\kappa_2)^2 - 2(1-\sigma)(\kappa_1\kappa_2 - \tau^2)\right] \\ &\quad = C(1-\sigma)(\kappa_2^2 + \tau^2) \text{......................(43)}. \end{aligned}$$

Hence the potential energy of the bent shell is

$$C(1-\sigma)a^2\int_0^{2\pi} d\phi \int_{\theta_0}^{\theta_1} \sin\theta d\theta\,(\kappa_2^2 + \tau^2) \text{.........(44)},$$

where θ_0 and θ_1 are the values of θ at the bounding-small circles.

On integration with respect to ϕ the products of different A's or B's, and of A's with B's, disappear, and we find for the energy

$$2\pi\frac{C(1-\sigma)}{a^2}\int_{\theta_0}^{\theta_1}\Sigma\,(n^3-n)^2(A_n^2\tan^{2n}\tfrac{1}{2}\theta + B_n^2\cot^{2n}\tfrac{1}{2}\theta)\operatorname{cosec}^3\theta d\theta$$

$$+ \text{ a similar term in } A_n'^2 \text{ and } B_n'^2.$$

In the particular case of a hemispherical shell we must take $B_n = 0$ to avoid infinite displacements at the pole of the hemisphere, and, restoring A_n', we have for the potential energy

$$\begin{aligned} &2\pi\frac{C(1-\sigma)}{a^2}\Sigma\,(A_n^2 + A_n'^2)(n^3-n)^2\int_0^{\frac{1}{2}\pi}\tan^{2n}\tfrac{1}{2}\theta\operatorname{cosec}^3\theta d\theta \\ &\quad = \tfrac{1}{2}\pi\frac{C(1-\sigma)}{a^2}\Sigma\,(A_n^2 + A_n'^2)(n^3-n)(2n^2-1) \text{.....(45)}. \end{aligned}$$

The occurrence of (n^3-n) as a factor in the nth term of κ_1, κ_2, τ and of the expression for the potential energy shews that $n=0$ and $n=1$ correspond to modes of non-extensional deformation without strain, and in fact it is easily verified that the corresponding displacements are rigid-body displacements.

The coefficient $C(1-\sigma)$ which occurs is $\frac{4}{3}\mu h^3$, and thus the potential energy of bending of a spherical shell depends on the rigidity and not on the resistance to compression of the material.

The formula for the potential energy enables us to solve statical problems relating to the deformation produced by given forces. Lord Rayleigh[1] gives two examples of the application of this method to the equilibrium of a hemispherical bowl as follows:—

1°. When the points $\theta=\frac{1}{2}\pi$, $\phi=\frac{1}{2}\pi$, and $\theta=\frac{1}{2}\pi$, $\phi=\frac{3}{2}\pi$ are joined by a string of tension F, all the coefficients in (38) vanish except $A_2, A_4, \ldots A_{2r} \ldots$, and

$$A_{2r}=\frac{2(-)^r a^2 F}{\pi C(1-\sigma)(4r^2-1)(8r^2-1)}.$$

2°. When the pole $\theta=0$ is the lowest point of the shell, and is supported, while a rod of weight W is laid symmetrically along the diameter through $\theta=\frac{1}{2}\pi$, $\phi=\frac{1}{2}\pi$, the same set of coefficients appear as in the last example, and

$$A_{2r}=\frac{(-)^{r-1} a^2 W}{\pi C(1-\sigma)\, 2r\,(4r^2-1)(8r^2-1)},$$

the weight of the bowl being neglected.

The verification of these results will serve as an exercise for the student.

[1] *Proc. Lond. Math. Soc.* XIII. 1882. The results are here slightly corrected. It may be noticed here that forces which produce displacements independent of ϕ cannot be treated by this method. For instance, a hemispherical shell supported by vertical forces at its rim and bent by its own weight is a case for which the method fails. In this and similar problems the deformation depends on extension of the middle-surface and the shell is practically rigid.

360. Non-extensional Vibrations[1].

For the free vibrations we require the kinetic energy. The expression for the kinetic energy per unit area is

$$\rho h\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right],$$

"rotatory inertia" being neglected; and, by using the formulæ (38), this is easily shewn to be

$$\pi\rho a^2h\int_{\theta_0}^{\theta_1}\Sigma\,(\dot{A}_n^2+\dot{A}_n'^2)\,\{2\sin^2\theta+(\cos\theta+n)^2\}\tan^{2n}\tfrac{1}{2}\theta\sin\theta d\theta$$

$$+\,\pi\rho a^2h\int_{\theta_0}^{\theta_1}\Sigma\,(\dot{B}_n^2+\dot{B}_n'^2)\,\{2\sin^2\theta+(\cos\theta+n)^2\}\cot^{2n}\tfrac{1}{2}\theta\sin\theta d\theta \quad (46),$$

where the dot denotes differentiation with respect to the time.

In the case of a hemispherical shell the B's must be omitted, and the limits of integration are 0 and $\frac{1}{2}\pi$; and, by writing $z=1+\cos\theta$, we find that the integral to be evaluated is

$$\int_1^2\{2z\,(2-z)+(n-1+z)^2\}\,(2-z)^n z^{-n}\,dz.$$

This is easily evaluated when n is given by expanding the binomial, and, if $f(n)$ be the value of the integral, the kinetic energy is

$$\pi\rho a^2h\Sigma\,(\dot{A}_n^2+\dot{A}_n'^2)\,f(n).$$

The kinetic and potential energies being reduced to sums of squares the coefficients A_n... are normal coordinates in the non-extensional vibrations.

The frequency of the nth component vibration is $p/2\pi$ where

$$p^2=\tfrac{2}{3}\,\frac{\mu h^2}{\rho a^4}\,\frac{(n^3-n)\,(2n^2-1)}{f(n)}\quad\ldots\ldots\ldots\ldots\ldots\ldots(47).$$

When the shell is not hemispherical but bounded by a small circle the formula is similar but $f(n)$ is different.

361. Discussion of non-extensional vibrations.

We have now to enquire whether the non-extensional displacements investigated in art. 358 can satisfy the general equations of free vibration, and we shall consider a particular type of displacements which correspond to a normal vibration under the condition that no line on the middle-surface is altered in length.

[1] Lord Rayleigh, *loc. cit.*

Let us assume that

$$\left.\begin{aligned} u &= A\sin\theta\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt},\\ v &= A\sin\theta\tan^n\tfrac{1}{2}\theta\sin n\phi\, e^{\iota pt},\\ w &= -A(\cos\theta+n)\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt}\end{aligned}\right\}\text{.........} (49).$$

These displacements make $\epsilon_1 = \epsilon_2 = \varpi = 0$, and

$$\left.\begin{aligned} -\kappa_1 = \kappa_2 &= \frac{n^3-n}{a^2}A\,\mathrm{cosec}^2\theta\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt},\\ \tau &= \frac{n^3-n}{a^2}A\,\mathrm{cosec}^2\theta\tan^n\tfrac{1}{2}\theta\sin n\phi\, e^{\iota pt},\end{aligned}\right\}\text{......}(50).$$

The stress-couples are given by the equations

$$\left.\begin{aligned} G_1 = G_2 &= C(1-\sigma)\frac{n^3-n}{a^2}A\,\mathrm{cosec}^2\theta\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt},\\ H &= C(1-\sigma)\frac{n^3-n}{a^2}A\,\mathrm{cosec}^2\theta\tan^n\tfrac{1}{2}\theta\sin n\phi\, e^{\iota pt}\end{aligned}\right\}\text{...}(51).$$

The stress-components T_1 and T_2 which act across the parallels and the meridians in the direction of the normal to the sphere are given by equations (1) of art. 343, and we have accordingly

$$\left.\begin{aligned} T_1 &= \frac{1}{a}\left[\frac{\partial G_1}{\partial\theta} + (G_1+G_2)\cot\theta - \frac{1}{\sin\theta}\frac{\partial H}{\partial\phi}\right],\\ T_2 &= -\frac{1}{a}\left[\frac{\partial H}{\partial\theta} + 2H\cot\theta + \frac{1}{\sin\theta}\frac{\partial G_2}{\partial\phi}\right]\end{aligned}\right\}\text{......}(52);$$

with the above values of G_1, G_2, and H it is easily shewn that T_1 and T_2 vanish identically.

Again, from equations (42) of art. 339 we find

$$P_1 = P_2 = \tfrac{1}{2}C\sigma R_0/\mu, \qquad U_1 = U_2 = 0\text{............}(53).$$

The equations of vibration (2) of art. 343 become

$$\left.\begin{aligned} -2\rho hp^2 A\sin\theta\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt} &= \frac{1}{a}\frac{\partial P_1}{\partial\theta},\\ -2\rho hp^2 A\sin\theta\tan^n\tfrac{1}{2}\theta\sin n\phi\, e^{\iota pt} &= \frac{1}{a\sin\theta}\frac{\partial P_1}{\partial\phi},\\ 2\rho hp^2 A(\cos\theta+n)\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt} &= -\frac{2P_1}{a}\end{aligned}\right\}\text{...}(54).$$

These equations are incompatible, and we conclude that the equations of vibration cannot be satisfied without taking account of the extension of the middle-surface.

362. Subsidiary displacements.

In order to take extension into account we assume that u, v, w differ from the values given for them in equations (49) by terms which are of the order h^2A/a^2. These give rise to extensions of the middle-surface which provide values of P_1, P_2, U_1, U_2 differing from those given in (53) above, but we may omit the additional terms which they contribute to G_1, G_2, H. The stresses T_1 and T_2 then continue to vanish. We may omit also the changes produced by the subsidiary displacements in the terms arising from the inertia of the shell, since p^2 is proportional to h^2. We thus obtain the equations

$$\left.\begin{aligned} &-2\rho hp^2A\sin\theta\tan^n\tfrac{1}{2}\theta\cos n\phi\, e^{\iota pt} \\ &\qquad=\frac{1}{a}\left[\frac{\partial P_1}{\partial\theta}+(P_1-P_2)\cot\theta+\frac{1}{\sin\theta}\frac{\partial U_2}{\partial\phi}\right], \\ &-2\rho hp^2A\sin\theta\tan^n\tfrac{1}{2}\theta\sin n\phi\, e^{\iota pt} \\ &\qquad=\frac{1}{a}\left[\frac{\partial U_1}{\partial\theta}+(U_1+U_2)\cot\theta+\frac{1}{\sin\theta}\frac{\partial P_2}{\partial\phi}\right], \\ &2\rho hp^2A(\cos\theta+n)\tan^n\tfrac{1}{2}\theta\cos n\phi e^{\iota pt}=-\frac{1}{a}(P_1+P_2) \end{aligned}\right\} \ldots(55).$$

In these equations (see art. 339),

$$\left.\begin{aligned} P_1&=\frac{3C}{h^2}(\epsilon_1+\sigma\epsilon_2)+\tfrac{1}{2}C\sigma\frac{R_0}{\mu}, \\ P_2&=\frac{3C}{h^2}(\epsilon_2+\sigma\epsilon_1)+\tfrac{1}{2}C\sigma\frac{R_0}{\mu}, \\ U_1=U_2&=\frac{3C}{2h^2}(1-\sigma)\varpi \end{aligned}\right\} \ldots\ldots\ldots\ldots (56);$$

where R_0 is unknown, but

$$\left.\begin{aligned} a\epsilon_1&=\frac{\partial \mathrm{u}}{\partial\theta}+\mathrm{w}, \\ a\epsilon_2&=\frac{1}{\sin\theta}\frac{\partial \mathrm{v}}{\partial\phi}+\mathrm{u}\cot\theta+\mathrm{w}, \\ a\varpi&=\frac{1}{\sin\theta}\frac{\partial \mathrm{u}}{\partial\phi}+\frac{\partial \mathrm{v}}{\partial\theta}-\mathrm{v}\cot\theta \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(57).$$

Now we shall put

$$P_1=L\cos n\phi\, e^{\iota pt},\quad P_2=M\cos n\phi\, e^{\iota pt}.\quad U_1=U_2=N\sin n\phi\, e^{\iota pt}\ldots(58),$$

and write
$$K=2\rho hap^2A\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(59).$$

Then the above equations (55) become

$$\left.\begin{aligned}\frac{dL}{d\theta}+(L-M)\cot\theta+\frac{n}{\sin\theta}N&=-K\sin\theta\tan^n\tfrac{1}{2}\theta,\\ \frac{dN}{d\theta}+2N\cot\theta-\frac{n}{\sin\theta}M&=-K\sin\theta\tan^n\tfrac{1}{2}\theta,\\ L+M&=-K(\cos\theta+n)\tan^n\tfrac{1}{2}\theta\end{aligned}\right\}\ldots(60).$$

Eliminating M we find

$$\left.\begin{aligned}&\sin\theta\frac{dL}{d\theta}+2L\cos\theta+nN+K[\sin^2\theta+\cos\theta(\cos\theta+n)]\tan^n\tfrac{1}{2}\theta=0,\\ &\sin\theta\frac{dN}{d\theta}+2N\cos\theta+nL+K[\sin^2\theta+n(\cos\theta+n)]\tan^n\tfrac{1}{2}\theta=0\end{aligned}\right\}\ (61).$$

By adding and subtracting these equations we find

$$\left.\begin{aligned}&\left(\sin\theta\frac{d}{d\theta}+2\cos\theta+n\right)(L+N)\\ &\qquad\qquad+K(1+2n\cos\theta+n^2+\sin^2\theta)\tan^n\tfrac{1}{2}\theta=0,\\ &\left(\sin\theta\frac{d}{d\theta}+2\cos\theta-n\right)(L-N)+K(\cos^2\theta-n^2)\tan^n\tfrac{1}{2}\theta=0.\end{aligned}\right\}$$

These are linear differential equations of the first order, and their integrating factors are respectively $\sin\theta\tan^n\frac{1}{2}\theta$ and $\sin\theta\cot^n\frac{1}{2}\theta$, so that we have

$$\left.\begin{aligned}&(L+N)\sin^2\theta\tan^n\tfrac{1}{2}\theta\\ &\qquad+K\textstyle\int(1+2n\cos\theta+n^2+\sin^2\theta)\sin\theta\tan^{2n}\tfrac{1}{2}\theta d\theta=\text{const.},\\ &(L-N)\sin^2\theta\cot^n\tfrac{1}{2}\theta+K\textstyle\int(\cos^2\theta-n^2)\sin\theta d\theta=\text{const.}\end{aligned}\right\}\ (62).$$

Taking, as in art. 360, the case where the pole $\theta=0$ is included and the pole $\theta=\pi$ is excluded from the shell, the constants are to be determined by the facts that $L+N$ does not become infinite when $\theta=0$, and that L vanishes at the free edge. Thus L and N and consequently P_1, P_2, and U_1 may be regarded as completely known.

Now equations (56) give us

$$\left.\begin{aligned}(\epsilon_1-\epsilon_2)&=\frac{h^2}{3C(1-\sigma)}(P_1-P_2),\\ \varpi&=\frac{2h^2}{3C(1-\sigma)}U_1\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ (63).$$

By (57) these become two equations for the determination of u and v, viz. we have

$$\left.\begin{aligned}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{\mathrm{u}}{\sin\theta}\right)-\frac{\partial}{\partial\phi}\left(\frac{\mathrm{v}}{\sin\theta}\right)&=\frac{h^2}{3Ca(1-\sigma)}(P_1-P_2),\\ \sin\theta\frac{\partial}{\partial\theta}\left(\frac{\mathrm{v}}{\sin\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{\mathrm{u}}{\sin\theta}\right)&=\frac{2h^2}{3Ca(1-\sigma)}U_1\end{aligned}\right\}\dots(64).$$

The solutions of these equations consist of complementary functions of the same form as (49) of art. 361 and particular integrals. To find the latter we take

$$\mathrm{u}=U\cos n\phi\, e^{\iota pt},\quad \mathrm{v}=V\sin n\phi\, e^{\iota pt},$$

then

$$\left.\begin{aligned}\sin\frac{dU}{d\theta}+U\cos\theta-nV&=\frac{h^2a}{3C(1-\sigma)}(L-M)\sin\theta,\\ \sin\theta\frac{dV}{d\theta}+V\cos\theta-nU&=\frac{2h^2a}{3C(1-\sigma)}N\sin\theta\end{aligned}\right\}\dots(65).$$

From these U and V may be determined in the same way as L and N were found from equations (61).

The subsidiary displacements $U\cos n\phi\, e^{\iota pt}$, $V\sin n\phi\, e^{\iota pt}$ are those required for the satisfaction of the differential equations of free vibration. No determination has been made of w, and no subsidiary displacement in the direction of the normal to the sphere is required. On the other hand the known values of P_1 and P_2 being substituted in equations (56) it is seen that any small value of w is associated with a certain small value of R_0.

It may be remarked that since L, M, N contain $2\rho hap^2A$ as a factor the expressions for U, V found from (65) will contain as a factor $\frac{2\rho h^3p^2a^2A}{3C(1-\sigma)}$, and with the value of p^2 given in art. 360 this will be $\frac{1}{3}\frac{h^2}{a^2}\frac{(n^3-n)(2n^2-1)}{f(n)}A$, where $f(n)$ is a certain rational function of n; so that the subsidiary displacements required to satisfy the differential equations are in fact of the order of the product of the non-extensional displacements and the square of the ratio (thickness : diameter).

In regard to the boundary-conditions it follows from the forms of G_1, H, T_1 given in art. 361 that the non-extensional

solutions are incapable by themselves of satisfying the boundary-conditions

$$G_1 = 0, \quad T_1 - \frac{1}{a \sin\theta}\frac{\partial H}{\partial \phi} = 0,$$

which hold at any small circle, nor will the subsidiary displacements just investigated enable us to satisfy these conditions. There can however be little doubt that by means of additional subsidiary displacements of the kind indicated in art. 349 these as well as the remaining conditions

$$P_1 = 0, \quad U_1 - H/a = 0$$

can be satisfied.

363. Extensional Vibrations.

We proceed now to consider the extensional vibrations of the spherical shell.

According to art. 345 we have the equations of vibration in the form

$$\left.\begin{aligned} 2\rho h \frac{\partial^2 \mathrm{u}}{\partial t^2} &= \frac{1}{a}\left[\frac{\partial P_1}{\partial \theta} + (P_1 - P_2)\cot\theta + \frac{1}{\sin\theta}\frac{\partial U_1}{\partial \phi}\right], \\ 2\rho h \frac{\partial^2 \mathrm{v}}{\partial t^2} &= \frac{1}{a}\left[\frac{\partial U_1}{\partial \theta} + 2U_1\cot\theta + \frac{1}{\sin\theta}\frac{\partial P_2}{\partial \phi}\right], \\ 2\rho h \frac{\partial^2 \mathrm{w}}{\partial t^2} &= -\frac{1}{a}\left[P_1 + P_2\right] \end{aligned}\right\} \ldots (66),$$

where

$$P_1 = \frac{3C}{h^2}(\epsilon_1 + \sigma\epsilon_2), \quad P_2 = \frac{3C}{h^2}(\epsilon_2 + \sigma\epsilon_1), \quad U_1 = \frac{3C(1-\sigma)}{2h^2}\varpi \ldots (67),$$

and ϵ_1, ϵ_2, and ϖ are given by (57).

Regarding especially the case of a complete sphere or a segment bounded by a small circle $\theta = \text{const.}$ we shall take u, v, w to be proportional to $e^{\iota pt}$, and u and w proportional to $\cos s\phi$, while v is proportional to $\sin s\phi$. Thus we write

$$\mathrm{u} = U\cos s\phi\, e^{\iota pt}, \quad \mathrm{v} = V\sin s\phi\, e^{\iota pt}, \quad \mathrm{w} = W\cos s\phi\, e^{\iota pt} \ldots (68),$$

where s is an integer and U, V, W are functions of θ only. Also we shall write

$$\kappa^2 = \tfrac{4}{3}\frac{p^2a^2\rho h^3}{C(1-\sigma)} = \frac{p^2a^2\rho}{\mu} \ldots\ldots\ldots\ldots\ldots\ldots (69).$$

Making these substitutions, the equations of vibration reduce to

$$\left.\begin{aligned}
&\frac{\partial}{\partial\theta}\left[\frac{\partial U}{\partial\theta}+W+\sigma\left(\frac{sV}{\sin\theta}+U\cot\theta+W\right)\right]\\
&\quad+(1-\sigma)\cot\theta\left(\frac{\partial U}{\partial\theta}-U\cot\theta-\frac{sV}{\sin\theta}\right)+\tfrac{1}{2}(1-\sigma)\left[s\frac{\partial}{\partial\theta}\left(\frac{V}{\sin\theta}\right)-\frac{s^2}{\sin^2\theta}U\right]\\
&\qquad\qquad+\tfrac{1}{2}\kappa^2(1-\sigma)\,U=0,\\
&\tfrac{1}{2}(1-\sigma)\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{V}{\sin\theta}\right)-\frac{sU}{\sin\theta}\right]\\
&\quad+(1-\sigma)\cos\theta\left[\frac{\partial}{\partial\theta}\left(\frac{V}{\sin\theta}\right)-\frac{sU}{\sin^2\theta}\right]\\
&\quad-\frac{s}{\sin\theta}\left[\frac{sV}{\sin\theta}+U\cot\theta+W+\sigma\left(\frac{\partial U}{\partial\theta}+W\right)\right]+\tfrac{1}{2}\kappa^2(1-\sigma)\,V=0,\\
&-(1+\sigma)\left[\frac{\partial U}{\partial\theta}+U\cot\theta+\frac{sV}{\sin\theta}+2W\right]+\tfrac{1}{2}\kappa^2(1-\sigma)\,W=0
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(70).$$

If now we write

$$\frac{1}{c}=\frac{1-\sigma}{1+\sigma}-\frac{4}{\kappa^2},\ \text{ or }\ \frac{2c}{\kappa^2}(1+\sigma)=\tfrac{1}{2}c(1-\sigma)-\tfrac{1}{2}(1+\sigma)\ldots(71),$$

the third of the above equations is

$$-\tfrac{1}{2}\frac{\kappa^2}{c}W+\left[\frac{\partial U}{\partial\theta}+U\cot\theta+\frac{sV}{\sin\theta}\right]=0\ldots\ldots\ldots(72).$$

Substituting for W in the first and second of (70), we find that $\frac{1}{2}(1-\sigma)$ is a factor that may be removed from both equations, and we obtain

$$\begin{aligned}
&\frac{\partial^2 U}{\partial\theta^2}+\cot\theta\frac{\partial U}{\partial\theta}+[2+\kappa^2-(1+s^2)\operatorname{cosec}^2\theta]\,U\\
&\qquad-\frac{2s\cos\theta}{\sin^2\theta}V+c\frac{\partial}{\partial\theta}\left(\frac{\partial U}{\partial\theta}+U\cot\theta+\frac{sV}{\sin\theta}\right)=0\ldots(73),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial^2 V}{\partial\theta^2}+\cot\theta\frac{\partial V}{\partial\theta}+[2+\kappa^2-\{1+s^2(1+c)\}\operatorname{cosec}^2\theta]\,V\\
&\qquad-(2+c)\frac{s\cos\theta}{\sin^2\theta}U-\frac{sc}{\sin\theta}\frac{\partial U}{\partial\theta}=0\ldots(74).
\end{aligned}$$

Differentiating (72) with respect to θ, and subtracting from (73), we have

$$(2+\kappa^2-s^2\operatorname{cosec}^2\theta)\,U-\frac{sV\cos\theta}{\sin^2\theta}-\frac{s}{\sin\theta}\frac{\partial V}{\partial\theta}+\tfrac{1}{2}\kappa^2\left(1+\frac{1}{c}\right)\frac{\partial W}{\partial\theta}=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(75).$$

364. Solution in Spherical Harmonics[1].

Equations (72), (74), and (75) are the equations to be satisfied by U, V, W. To solve them we take

$$x = U \sin\theta, \quad y = V \sin\theta \quad\text{.................(76)};$$

we first eliminate x by means of equation (75) which is

$$x = \frac{-\tfrac{1}{2}\kappa^2\left(1+\dfrac{1}{c}\right)\sin^3\theta\dfrac{\partial W}{\partial\theta} + s\,.\sin\theta\dfrac{\partial y}{\partial\theta}}{(2+\kappa^2)\sin^2\theta - s^2} \quad\text{.........(77)},$$

and arrive at two equations containing y and W. It will be found that these equations contain the same function of y which is

$$[(2+\kappa^2)\sin^2\theta - s^2]\left[\frac{\partial^2 y}{\partial\theta^2} - \cot\theta\frac{\partial y}{\partial\theta} + \{(2+\kappa^2)\sin^2\theta - s^2\}\frac{y}{\sin^2\theta}\right] - 2s^2\cot\theta\frac{\partial y}{\partial\theta},$$

so that it is easy to eliminate y, and the resulting equation for W turns out to be

$$\frac{\partial^2 W}{\partial\theta^2} + \cot\theta\frac{\partial W}{\partial\theta} + \left(\frac{2+\kappa^2}{1+c} - \frac{s^2}{\sin^2\theta}\right)W = 0 \quad\text{.........(78)}.$$

Again, writing (72) in the form

$$sy = \frac{\kappa^2}{2c}W\sin^2\theta - \sin\theta\frac{\partial x}{\partial\theta} \quad\text{..................(79)},$$

we can easily eliminate y from (74), and thus obtain an equation which turns out to be

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial x}{\partial\theta}\right) + \frac{x}{\sin\theta}[(2+\kappa^2)\sin^2\theta - s^2] + \tfrac{1}{2}\kappa^2\sin\theta\left(\frac{\partial W}{\partial\theta} - 2\cos\theta\frac{W}{c}\right) = 0 \quad\text{.........(80)}.$$

Equation (78) determines W, equation (80) determines x when W is known, and equation (79) determines y when x and W are known.

[1] Only the leading steps of the Analysis are given. The process was suggested by M. Mathieu's work in the *Journal de l'École Polytechn.*, Cahier 51, 1883.

Writing μ for $\cos\theta$, these equations become

$$\left.\begin{aligned} &\frac{d}{d\mu}\left[(1-\mu^2)\frac{dW}{d\mu}\right]+\left(\frac{2+\kappa^2}{1+c}-\frac{s^2}{1-\mu^2}\right)W=0,\\ &\frac{d}{d\mu}\left[(1-\mu^2)\frac{dx}{d\mu}\right]+\left(2+\kappa^2-\frac{s^2}{1-\mu^2}\right)c\\ &\qquad\qquad=\kappa^2\left[\frac{\mu}{c}W+\tfrac{1}{2}(1-\mu^2)\frac{dW}{d\mu}\right]=0,\\ &\qquad sy=(1-\mu^2)\left[\frac{\kappa^2}{2c}W+\frac{dx}{d\mu}\right] \end{aligned}\right\}\ldots(81).$$

The equation for W is the equation of tesseral harmonics of argument μ, rank s, and index n, where

$$n(n+1)=(2+\kappa^2)/(1+c)\ldots\ldots\ldots\ldots\ldots\ldots(82).$$

The solution of the equation for x consists of two parts. The first part is a particular integral depending on W and it can be shewn without difficulty that

$$\frac{\kappa^2}{2+\kappa^2}\,\frac{1+c}{2c}(1-\mu^2)\frac{dW}{d\mu}$$

is such a particular integral. The second part is a complementary function which satisfies the equation derived from the second of (81) by putting $W=0$, and this function is a tesseral harmonic of argument μ, rank s, and index m where

$$m(m+1)=2+\kappa^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(83).$$

We have thus the complete solutions for a shell including the pole $\mu=1$ in the forms:

$$\left.\begin{aligned} &W=A_{ns}T_n^{(s)}(\mu),\\ &x=B_{ms}T_m^{(s)}(\mu)+\frac{\kappa^2}{2+\kappa^2}\,\frac{1+c}{2c}(1-\mu^2)A_{ns}\frac{d}{d\mu}T_n^{(s)}(\mu),\\ &y=\frac{1-\mu^2}{s}\left[\frac{\kappa^2}{2c}W+\frac{dx}{d\mu}\right] \end{aligned}\right\}\ldots(84).$$

where $T_n^{(s)}(\mu)$ is that particular integral of the equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dT}{d\mu}\right]+\left[n(n+1)-\frac{s^2}{1-\mu^2}\right]T=0\ldots..(85),$$

which does not become infinite when $\mu=1$.

365. Complete spherical shell.

Let us suppose the spherical shell complete, so that the pole $\mu=-1$ is also included, then we know from the theory of Spherical Harmonics that $T_n^{(s)}(\mu)$ becomes infinite when $\mu=-1$ unless n is an integer. We thus find two classes of types of vibration.

In the vibrations of the first class $A_{ns}=0$ while m is an integer. The motion is purely tangential. The frequency is given by the equation

$$2+\kappa^2=m(m+1)$$

or

$$p^2=\frac{\mu}{a^2\rho}(m-1)(m+2) \quad\text{..................(86).}$$

The forms of the displacements when any normal vibration of this type is being executed are

$$\left.\begin{aligned} \mathrm{u} &= \frac{1}{\sqrt{(1-\mu^2)}}B_{ms}T_m^{(s)}(\mu)\begin{matrix}\cos s\phi\\ -\sin s\phi\end{matrix}\, e^{\iota pt}, \\ \mathrm{v} &= \sqrt{(1-\mu^2)}\frac{B_{ms}}{s}\frac{d}{d\mu}T_m^{(s)}(\mu)\begin{matrix}\sin s\phi\\ \cos s\phi\end{matrix}\, e^{\iota pt} \end{aligned}\right\} \quad\text{..........(87),}$$

while w vanishes.

These correspond to the purely tangential vibrations of a sphere or spherical shell of finite thickness investigated in I. ch. XI.

In the vibrations of the second class $B_{ms}=0$, while n is an integer. The motion is partly tangential and partly radial. The frequency is given by the equation

$$\frac{2+\kappa^2}{1+c}=n(n+1),$$

or from (71)

$$\kappa^4(1-\sigma)-2\kappa^2[1+3\sigma+n(n+1)]+4(n-1)(n+2)(1+\sigma)=0 \quad\text{......................(88),}$$

where $\kappa^2=p^2a^2\rho/\mu$,[1] as in equation (69).

The forms of the displacements when any normal vibration of this type is being executed are

[1] μ here is the rigidity, it can hardly be confused with $\mu=\cos\theta$ in $T_n^{(s)}\mu$.

$$\left.\begin{aligned}
u &= \frac{\kappa^2}{2n(n+1)c}\sqrt{(1-\mu^2)}\,A_{ns}\frac{d}{d\mu}T_n^{(s)}(\mu)\begin{matrix}\cos s\phi\\ -\sin s\phi\end{matrix}\,e^{\iota pt},\\
v &= \frac{\kappa^2}{2n(n+1)c}\frac{s}{\sqrt{(1-\mu^2)}}A_{ns}T_n^{(s)}(\mu)\begin{matrix}\sin s\phi\\ \cos s\phi\end{matrix}\,e^{\iota pt},\\
w &= A_{ns}T_n^{(s)}(\mu)\begin{matrix}\cos s\phi\\ -\sin s\phi\end{matrix}\,e^{\iota pt}
\end{aligned}\right\}\text{...(89),}$$

in which κ^2 is one of the roots of the equation (88), and p is the corresponding value found from (69), while

$$\frac{\kappa^2}{2c}=\frac{\kappa^2}{2}\frac{1-\sigma}{1+\sigma}-2.$$

366. Particular cases[1].

We may notice especially the purely radial vibrations, the modes that correspond to $n=1$, and the symmetrical modes for which $s=0$.

1°. Radial vibrations.

These are easily investigated from the general equations (66). We have to suppose that u and v vanish, and then it will be found that w must be independent of θ and ϕ, and that the frequency is given by the equation[2]

$$p^2=4\frac{1+\sigma}{1-\sigma}\frac{\mu}{\rho a^2}\text{........................(90).}$$

2°. Symmetrical modes for which $n=1$.

These are partly radial and partly tangential, but the displacement of any point takes place in the meridian plane.

The frequency is given by the equation

$$p^2=6\frac{1+\sigma}{1-\sigma}\frac{\mu}{\rho a^2}\text{........................(91),}$$

and the displacements along the meridian and the normal have the forms

$$u=\tfrac{1}{2}A_1\sin\theta\,e^{\iota pt},\quad w=A_1\cos\theta\,e^{\iota pt}\text{ ....(92).}$$

[1] The results only are given and the verification is left to the reader.

[2] Cf. I. art. 129.

3°. Unsymmetrical modes for which $n=1$.

For these modes $s=1$. The frequency is given by (91), and the displacements have the forms

$$\left.\begin{aligned} u &= -\tfrac{1}{2}A_1\cos\theta \begin{matrix}\cos\phi\\ -\sin\phi\end{matrix}\, e^{\iota pt},\\ v &= \tfrac{1}{2}A_1 \begin{matrix}\sin\phi\\ \cos\phi\end{matrix}\, e^{\iota pt},\\ w &= A_1\sin\theta \begin{matrix}\cos\phi\\ -\sin\phi\end{matrix}\, e^{\iota pt}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots (93).$$

4°. Purely tangential modes for which $s=0$.

It will be found that u and w vanish while v is given by the equation

$$v = B_n \frac{d}{d\theta}\{P_n(\cos\theta)\}\, e^{\iota pt},$$

where P_n denotes Legendre's nth coefficient, and the frequency is given by the equation

$$p^2 = (n-1)(n+2)\frac{\mu}{\rho a^2} \ldots\ldots\ldots\ldots\ldots\ldots (94).$$

5°. Partly radial modes for which $s=0$.

The displacement of any point takes place in the meridian plane, and the component displacements along the meridian and the normal have the forms:

$$\left.\begin{aligned} u &= -\frac{1}{n(n+1)}\left(\frac{\kappa^2}{2}\frac{1-\sigma}{1+\sigma}-2\right)A_n\frac{d}{d\theta}\{P_n(\cos\theta)\}\, e^{\iota pt},\\ w &= A_n P_n(\cos\theta)\, e^{\iota pt}\end{aligned}\right\}\ldots(95),$$

and the frequency is given by $p^2=\kappa^2\mu/\rho a^2$ where κ^2 is a root of the equation (88). There are two different periods with similar types of motion.

367. Shell bounded by small circle.

When the spherical shell is bounded by a small circle we have to satisfy the boundary-conditions

$$\epsilon_1 + \sigma\epsilon_2 = 0, \quad \varpi = 0,$$

or

$$\left.\begin{aligned} \frac{\partial U}{\partial\theta} + W + \sigma\left(\frac{sV}{\sin\theta} + U\cot\theta + W\right) &= 0,\\ \sin\theta\frac{\partial}{\partial\theta}\left(\frac{V}{\sin\theta}\right) - \frac{sU}{\sin\theta} &= 0\end{aligned}\right\}\ldots\ldots\ldots (96),$$

for a particular value of θ.

The substitution of the values of U, V, W from (84) by means of (76) would lead to two equations from which the constants A_{ns} and B_{ms} could be eliminated, and there would remain an equation connecting $T_n^{(s)}(\mu)$, $T_m^{(s)}(\mu)$ to be satisfied for a particular value of μ. Since n and m are known in terms of κ^2, this is really an equation to find κ^2, but the method of solving it is unknown.

The particular case of the symmetrical vibrations of a hemispherical shell can however be worked out, and it can be shewn that they are included in the cases 2°, 4°, and 5° of the last article in which further n must be an odd integer. This case is left to the reader, but the work is given in my paper (*Phil. Trans.* A. 1888).

368. Equilibrium under extension.

We have seen in the footnote on p. 273 that some modes of displacement of a spherical shell are possible in which the potential energy of bending vanishes and the strain produced by some applications of force must be extensional. If in fact u, v, w can be independent of ϕ or proportional to sines and cosines of ϕ the corresponding forces tend to stretch the middle-surface.

When the shell is subject to forces $2Xh$, $2Yh$, $2Zh$ per unit area parallel to the tangent to the meridian, the tangent to the parallel, and the normal, and X, Y, Z are independent of ϕ, the displacements u, v, w can be independent of ϕ, and the equations of equilibrium (45) of art. 341 become

$$\left.\begin{aligned} &2Xh + \frac{1}{a}\left[\frac{\partial P_1}{\partial \theta} + (P_1 - P_2)\cot\theta\right] = 0,\\ &2Yh + \frac{1}{a}\left[\frac{\partial U_1}{\partial \theta} + 2U_1\cot\theta\right] = 0,\\ &2Zh - \frac{1}{a}(P_1 + P_2) = 0 \end{aligned}\right\} \quad \text{.........} \ (97),$$

in which

$$\left.\begin{aligned} &P_1 = \frac{3C}{ah^2}\left[\frac{\partial \mathrm{u}}{\partial \theta} + \mathrm{w} + \sigma(\mathrm{u}\cot\theta + \mathrm{w})\right],\\ &P_2 = \frac{3C}{ah^2}\left[\mathrm{u}\cot\theta + \mathrm{w} + \sigma\left(\frac{\partial \mathrm{u}}{\partial \theta} + \mathrm{w}\right)\right],\\ &U_1 = \frac{3C(1-\sigma)}{2ah^2}\sin\theta\frac{\partial}{\partial \theta}\left(\frac{\mathrm{v}}{\sin\theta}\right) \end{aligned}\right\} \quad \text{.........} \ (98).$$

Now if we write

$$X' = a^2 X/\mu = \tfrac{4}{3} a^2 h^3 X/C(1-\sigma) \quad \text{...............(99)},$$

and similarly for Y' and Z' in terms of Y and Z, we shall find equations[1] which reduce to

$$\left.\begin{aligned} &\frac{\partial^2 \mathrm{u}}{\partial\theta^2} + \cot\theta \frac{\partial \mathrm{u}}{\partial\theta} + \mathrm{u}(2 - \operatorname{cosec}^2\theta) \\ &\qquad\qquad + \frac{1+\sigma}{1-\sigma}\frac{\partial}{\partial\theta}\left[\frac{\partial \mathrm{u}}{\partial\theta} + \mathrm{u}\cot\theta + 2\mathrm{w}\right] + X' = 0, \\ &\frac{\partial^2 \mathrm{v}}{\partial\theta^2} + \cot\theta\frac{\partial \mathrm{v}}{\partial\theta} + \mathrm{v}(2 - \operatorname{cosec}^2\theta) + Y' = 0, \\ &\frac{1+\sigma}{1-\sigma}\left[\frac{\partial \mathrm{u}}{\partial\theta} + \mathrm{u}\cot\theta + 2\mathrm{w}\right] - \tfrac{1}{2}Z' = 0, \end{aligned}\right\} \text{...(100)}.$$

The first two become, by using the third,

$$\begin{aligned} &\frac{\partial^2 \mathrm{u}}{\partial\theta^2} + \cot\theta\frac{\partial \mathrm{u}}{\partial\theta} + \mathrm{u}(2 - \operatorname{cosec}^2\theta) + X' + \tfrac{1}{2}\frac{\partial Z'}{\partial\theta} = 0, \\ &\frac{\partial^2 \mathrm{v}}{\partial\theta^2} + \cot\theta\frac{\partial \mathrm{v}}{\partial\theta} + \mathrm{v}(2 - \operatorname{cosec}^2\theta) + Y' \qquad\qquad = 0. \end{aligned}$$

A particular integral when X', Y', Z' vanish is $\sin\theta$, and thus we can find the complete primitives in the forms

$$\left.\begin{aligned} \mathrm{u} &= A_1\sin\theta + B_1(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) \\ &\qquad - \sin\theta\int\left\{\operatorname{cosec}^3\theta\int\sin^2\theta\left(X' + \tfrac{1}{2}\frac{\partial Z'}{\partial\theta}\right)d\theta\right\}d\theta, \\ \mathrm{v} &= A_2\sin\theta + B_2(\cot\theta - \sin\theta\log\tan\tfrac{1}{2}\theta) \\ &\qquad\qquad - \sin\theta\int\{\cos^3\theta\int\sin^2\theta\, Y'd\theta\}\,d\theta \end{aligned}\right\} \text{...(101)},$$

and w is given by the third of (100).

The application to the case of a hemispherical bowl supported by uniform tensions at its rim, and deflected by its own weight is given with other applications in a paper by the author[2]. In the case referred to it is shewn that v vanishes, and

$$\left.\begin{aligned} \mathrm{u} &= \frac{g\rho a^2}{2\mu}[-\sin\theta + \tan\tfrac{1}{2}\theta - \log(1+\cos\theta)], \\ \mathrm{w} &= \frac{g\rho a^2}{2\mu}[2\cos\theta - \tfrac{1}{2} + \cos\theta\log(1+\cos\theta)], \end{aligned}\right\}$$

[1] Only the leading steps of the analysis are given in this article.

[2] 'On the Equilibrium of a Thin Elastic Spherical Bowl'. *Proc. Lond. Math. Soc.* xx. 1889.

where ρ is the density, and μ the rigidity of the material of the shell.

The investigation of the solutions in which

$$u \propto \cos \phi, \qquad v \propto \sin \phi, \qquad w \propto \cos \phi$$

is given in the paper mentioned above, but the interest is mainly analytical. The general case there investigated where $u \propto \cos s\phi \ldots$ is entirely unimportant as, whenever the forces in operation could produce displacements of this type, the displacements would be non-extensional, and the method of art. 359 would be applicable. The difficulty of satisfying the equations exactly would be evaded by the method of arts. 348 and 362.

CHAPTER XXIII.

STABILITY OF ELASTIC SYSTEMS.

369. THE part of our subject upon which we are now entering is among the most difficult for theoretical treatment and the most important for practical purposes. We propose in the first place to lay down the principles upon which the discussion of any case of buckling or collapse must be founded. We shall then proceed to illustrate the application of these principles to various problems in which configurations of equilibrium of thin rods, plates, or shells may become unstable.

370. Possible Instability confined to thin rods, plates, and shells[1].

We remark in the first place that a configuration of equilibrium of an elastic solid body cannot be unstable if it be the only configuration, and consequently the possibility of instability is bound up with the failure of the theorem of uniqueness of solution given in (ϵ) of I. art. 66. We have therefore in the first place to examine under what conditions two solutions of the equations of elastic equilibrium of the same body under the same system of forces can exist.

Consider an elastic solid body all whose dimensions are of the same order of magnitude, and suppose some such conditions as those in (γ) of I. art. 66 imposed, in order to prevent purely rigid-body displacements; then, in any strained condition, the displacement of every particle from its unstrained position must be very

[1] For an analytical discussion of the theorem of this article and exhibition of the criterion in art. 371 the reader is referred to Mr G. H. Bryan's paper 'On the Stability of Elastic Systems', *Proc. Camb. Phil. Soc.* VI. 1888.

small. Suppose that the body is held in a strained position by given forces applied to given particles of it in directions fixed in space. The points of application of the forces and their directions with reference to line-elements fixed in the body will after strain be very nearly the same as before strain, and it will make only a very small difference to the equations of equilibrium or the boundary-conditions that the given forces are actually applied to the system in the strained state. In the equations and conditions used in vol. I. it is assumed that this difference may be neglected, and whenever this assumption is justified the theorem of uniqueness of solution holds good.

Consider next a body in which finite displacements can be accompanied by only infinitesimal strains. The difference between the equations that hold when the displacements of every point are infinitesimal and those which hold when the displacements are finite is very considerable. The fact that the forces are applied in a configuration finitely different from the natural state changes altogether the character both of the equations of equilibrium and of the boundary-conditions; and the reasoning by which the theorem of uniqueness of solution was established has no reference to such a state of things.

We conclude that more than one configuration of equilibrium is impossible unless some displacement may be finite. Rejecting rigid-body displacements, we deduce that the only bodies for which any configuration of elastic equilibrium can be unstable are very thin rods, plates, and shells.

371. Criterion of Stability.

When two configurations of equilibrium are possible the test by which we distinguish which of them is stable is that in the stable configuration the potential energy is a minimum. Now it is clear from the formulæ that have been given for the potential energy of thin rods, plates, or shells that in any non-extensional configuration the energy can be made as small as we please in comparison with that in any extensional configuration by sufficiently diminishing those linear dimensions of the body which are supposed small. It follows that whenever two configurations are possible, of which one involves extension and the other no extension, the former is an unstable configuration. The like holds good also for configurations which are only approximately non-extensional,

as in the case of thin shells, where the extension of the middle surface is of the order of the product of the square of the thickness and the change of curvature, or where the extension becomes important only at points close to the boundaries.

372. Methods of Investigation.

There are two methods by which the stability of a given configuration of equilibrium may be investigated. In the first method the other possible configurations of equilibrium are known, and we have only to determine whether the potential energy is a minimum. In the second method one extensional configuration is known, and its stability is to be discussed, and the other possible configurations are unknown and cannot easily be found. The known extensional configuration differs from the unstrained state only by displacements of the order that could occur in a body all whose dimensions are finite. The unknown configurations of equilibrium with non-extensional displacements are in general finitely different from the unstrained state. But even in a thin rod, plate, or shell, under forces which tend to produce extension, equilibrium without extension cannot occur unless a certain relation of inequality connecting the constants of the system be satisfied. Suppose l is a particular constant of the system, (*e.g.* the length of a loaded column,) and suppose that equilibrium without extension is impossible unless $l > l_0$; then the extensional configuration becomes unstable when this relation of inequality is satisfied. Conversely, if the extensional configuration become unstable when $l > l_0$, then when $l - l_0$ is indefinitely small there is a configuration of equilibrium without extension differing indefinitely little from the extensional one[1]. To discover this relation of inequality, therefore, we seek a configuration of equilibrium without extension in which the displacements are infinitesimal, proceeding from the equations of finite displacement, and passing to a limit. The method of doing this will be illustrated below in several important examples.

373. Stability of Thin Rod loaded vertically[2].

We proceed to consider the very important practical problem of the conditions of buckling of a thin rod one end of which is fixed while the other end supports a weight. We shall suppose a

[1] Cf. Poincaré, *Acta Mathematica*, VII. 1885, pp. 261 sq.

[2] Euler's problem (see Introduction).

rod of length l built-in vertically at its lower end, and loaded at its upper end with a weight W. All the possible configurations of equilibrium of the system are known, for either the elastic central-line is straight and is compressed in a definite ratio[1], or it is one of the curves of the *elastica* family. The conditions of the problem give rise to a relation of inequality which must hold if the strained elastic central-line is not straight. The loaded end being subject to force alone without couple must be an inflexion on the elastica, and the tangent at the fixed end being parallel to the line of action of the load, the length of the rod must be either half the arc between consecutive inflexions, or an odd multiple of this; so that the arc between consecutive inflexions is $2l/(2n+1)$, where n is an integer or zero. Now we have shewn in art. 228 that the load W, the arc $2l/(2n+1)$ between consecutive inflexions, and the real quarter period K of elliptic functions by which the curve is defined are connected by the relation

$$\frac{2l}{2n+1}\sqrt{\frac{W}{B}} = 2K \qquad (1),$$

where B is the flexural rigidity of the rod.

Since K is not less than $\frac{1}{2}\pi$, it follows that there is no elastica which satisfies the conditions of the problem if

$$Wl^2 < \tfrac{1}{4}\pi^2 B \qquad (2).$$

We conclude that the straight form is the only possible form when l is $< \frac{1}{2}\pi\sqrt{(B/W)}$, or when $W < \frac{1}{4}\pi^2 B/l^2$, but when these limits are exceeded the straight form is unstable (fig. 55 *a*).

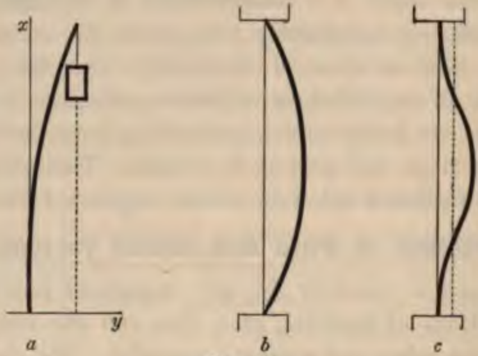


Fig. 55.

[1] With the notation of our chapters on rods this ratio is $W/E\omega$.

In like manner when the rod is pressed between supports so that both ends are free to change their directions, both ends are inflexions on any possible *elastica*, and the straight form is unstable if

$$Wl^2 > \pi^2 B \text{..........................(3)},$$

where W is the pressure at either end, and l the length of the rod (fig. 55 b).

Again, if the rod be built-in vertically at both ends, and the upper end which is loaded be constrained to remain vertically over the lower end[1], the central-line if curved must cut the line of inflexions twice, and each end must be at the maximum distance from this line. Hence the condition of instability of the straight form is

$$Wl^2 > 4\pi^2 B \text{..........................(4)},$$

where W is the load, and l the length of the rod (fig. 55 c).

374. Application of the second method of investigation.

The condition (2) found in the preceding article may be obtained without using the properties of the elastica.

For suppose if possible that the rod is very slightly bent by the weight W without compression. Let axes of x and y be drawn through the fixed end (as in fig. 55 a) the axis x being vertical and the axis y directed towards that side to which the weight inclines. Then y is very small at all points of the elastic central-line, and the differential equation for this line is found by taking moments about any point for the equilibrium of the part between that point and the loaded end. The equation is

$$B\frac{d^2y}{dx^2} = W(y_0 - y) \text{......................(5)},$$

where y_0 is the value of y at the loaded end.

The solution of this equation for which y and dy/dx vanish with x, while $y = y_0$ when $x = l$, is

$$y = y_0\left[1 - \frac{\sin\left\{\sqrt{\frac{W}{B}}(l-x)\right\}}{\sin\left(\sqrt{\frac{W}{B}}\,l\right)}\right],$$

[1] These conditions apply more nearly than those of the preceding paragraphs to a slender pillar supporting a roof, but in any practical problem the load W would probably be difficult to estimate exactly.

provided $$\cos l\sqrt{\frac{W}{B}} = 0.$$

Hence there is a solution of the problem in the form proposed when

$$l\sqrt{(W/B)} = \tfrac{1}{2}(2n-1)\pi,$$

where n is any integer, and the least value of l or W for which instability of the straight form occurs is given by the equation

$$Wl^2 = \tfrac{1}{4}\pi^2 B.$$

275. Stability of Inflexional Elastica.

Proceeding now with the theory of the rod supporting a weight, and taking the part between consecutive inflexions as $2l$, we observe that, in any possible configuration of equilibrium with displacements of bending only, when one end is built-in vertically, the central-line forms an odd number of half-waves of some curve of the elastica family, and a relation such as (7) connects the constants of the system with the form of the curve. When n is an integer such that

$$\tfrac{1}{2}(2n-1)\pi < l\sqrt{(W/B)} < \tfrac{1}{2}(2n+1)\pi \quad \text{........(9)}$$

n forms besides the unstable straight form are possible, and they consist respectively of 1, 3, ... $2n-1$ half-waves of different curves of the elastica family. The forms of these curves are respectively given by the equations

$$l = K\sqrt{\frac{B}{W}}, \quad l = 3K\sqrt{\frac{B}{W}}, \quad \ldots \quad l = (2n-1)K\sqrt{\frac{B}{W}} \ldots(10)$$

We wish to investigate the stability of these forms.

For this purpose we have to obtain an expression for the energy of the system in any configuration of equilibrium.

Let θ be the angle which the tangent to the elastica at any point makes with a vertical line drawn upwards, and α the value of θ at an inflexion, then $\sin\frac{1}{2}\alpha$ is the modulus k of the elliptic functions by which the curve is defined. Let ds be the element of arc of the curve, the potential energy of the system is

$$\tfrac{1}{2}B\int_0^l \left(\frac{d\theta}{ds}\right)^2 ds - W\int_0^l \cos\theta\, ds,$$

where the first term is the potential energy of bending, and the second is the potential energy of the raised weight.

Now equation (63) of art. 228 is

$$\tfrac{1}{2}B\left(\frac{d\phi}{ds}\right)^2 = W(\cos\phi - \cos\alpha),$$

so that the expression for the potential energy is

$$W\int_0^l (2\cos\phi - \cos\alpha)\,ds,$$

or

$$W(2h - l\cos\alpha) \text{ .... } (8),$$

where h is the height of the loaded end above the point of support.

When the curve consists of $2r+1$ half-bays of an elastica, the quantity h is $2r+1$ times the breadth of a half-bay, and the quantity l is $2r+1$ times the length of the arc of a half-bay. Now we have shewn in art. 228 that the breadth of a half-bay and the length of the arc of a half-bay are respectively

$$(2E-K)\sqrt{(B/W)} \text{ and } K\sqrt{(B/W)},$$

where E is the complete elliptic integral of the second kind with modulus k.

Hence the potential energy of the configuration with $2r+1$ half-bays is

$$(2r+1)\sqrt{(BW)}\{4E - 2K - K\cos\alpha\},$$

or

$$(2r+1)\sqrt{(BW)}\{4E - 3K + 2Kk^2\} \text{............} (9).$$

We shall now shew[1] that the potential energy increases as the number of half-bays increases. For let E_1, K_1, k_1, k_1' refer to the form with $2r+1$ half-bays, and E_2, K_2, k_2, k_2' to the form with $2s+1$ half-bays, and suppose $r > s$. Since the length is the same we have

$$(2r+1)K_1 = (2s+1)K_2.$$

The potential energy in the form with $2r+1$ half-bays is the greater if

$$(2r+1)(2E_1 + K_1k_1^2) > (2s+1)(2E_2 + K_2k_2^2),$$

[1] In the demonstration we use the relations

$$E = k'^2\left(K + k\frac{dK}{dk}\right) \text{ and } kk'^2\frac{d^2K}{dk^2} + (1-3k^2)\frac{dK}{dk} - kK = 0,$$

(where k' is the complementary modulus given by $k^2 + k'^2 = 1$), for which see Cayley's *Elliptic Functions*, pp. 48, 50.

i.e. if

$$(2r+1)\,k_1'^2\left(K_1+2k_1\frac{dK_1}{dk_1}\right)>(2s+1)\,k_2'^2\left(K_2+2k_2\frac{dK_2}{dk_2}\right),$$

i.e. if $$k_1'^2\left(1+\frac{2k_1}{K_1}\frac{dK_1}{dk_1}\right)>k_2'^2\left(1+\frac{2k_2}{K_2}\frac{dK_2}{dk_2}\right).$$

Now $$\frac{d}{dk}\left[k'^2\left(1+\frac{2k}{K}\frac{dK}{dk}\right)\right]=-\frac{2kk'^2}{K^2}\left(\frac{dK}{dk}\right)^2,$$

so that the quantity $k'^2\left(1+\dfrac{2k}{K}\dfrac{dK}{dk}\right)$ diminishes as k or K increases. This proves the theorem.

In the sequence of equilibrium configurations with 1, 3,... $2n-1$ half-bays possible under the condition (6) the potential energy constantly increases. It follows that the configuration with a single half-bay is the only one that can be stable for all displacements.

This conclusion may with advantage be illustrated by taking the particular case where two forms besides the unstable straight form just become possible, for which

$$\tfrac{3}{2}\pi - l\sqrt{(W/B)}$$

is a small positive quantity. The value of α for the form with one half-bay is given approximately by the equation

$$K=\tfrac{3}{2}\pi,$$

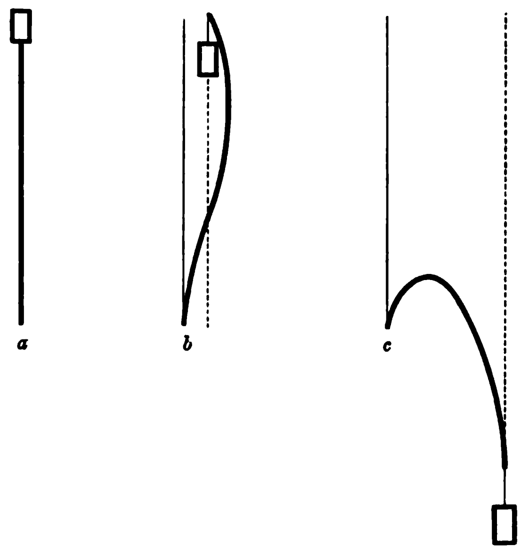


Fig. 56.

and the root α lies between 175^0 and 176^0. The value of α for the form with three half-bays is very small. The three configurations of equilibrium are shewn in the figure.

The form c with a single half-bay is stable. The potential energy is a maximum in the form a and a minimum in the form c. For the intermediate form it is stationary but it is not a minimum for all small displacements[1].

376. Height consistent with Stability.

In art. 373 we considered the problem of the greatest length of a vertical rod consistent with stability of the straight form, when the rod was supposed weightless and under the action of a given load; but it is clear that if the rod be very long its own weight may be considerable although its thickness is very small; and we shall now investigate the greatest length, consistent with stability, of a rod of given uniform line-mass, whose lower end is built-in vertically and whose upper end is free.

The straight form in which the strain is purely extensional will become unstable if there can be a configuration of equilibrium in which the rod is held slightly bent without stretching by the action of its own weight.

Take axes of x and y as in the figure, and let W be the weight of the rod, l its length, B its flexural rigidity, and y_0 the horizontal displacement of the free end, and take moments about any point (x, y) on the central-line for the equilibrium of the part between that point and the free end; we thus find the equation

$$B\frac{d^2y}{dx^2} = \int_x^l \frac{W}{l}(y' - y)\,dx' \qquad \text{.........(10)},$$

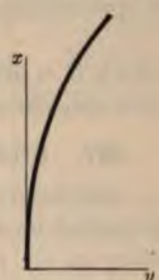


Fig. 57.

(x', y') being any point between (x, y) and the free end. Differentiating this equation with respect to x, and writing p for dy/dx, we find

[1] Prof. L. Saalschütz in his treatise *Der Belastete Stab* (Leipzig, 1880) has shewn that the form b is stable for all displacements in which the curve continues to be an elastica and the loaded end an inflexion. This result he arrives at by a consideration of the direction of the additional force required to hold the rod in the form of an elastica differing very little from b. This does not shew however that the form b is stable for all displacements, for there will be some elasticas differing very little from b for which the loaded end is not an inflexion. The student will find it an interesting exercise to verify Prof. Saalschütz's result by using the energy-criterion of stability.

$$B\frac{d^2p}{dx^2} = -\frac{W}{l}(l-x)\,p \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

which is the differential equation of the central-line. The terminal conditions are that y and p vanish with x, and dp/dx vanishes when $x = l$.

The differential equation (11) may be transformed to Bessel's equation by the substitutions

$$p = z\sqrt{(l-x)}, \qquad r^2 = (l-x)^3,$$

and it takes the form

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} + \left(\frac{4}{9}\frac{W}{lB} - \frac{1}{9}\frac{1}{r^2}\right)z = 0;$$

and the solution is of the form

$$z = AJ_{\frac{1}{3}}(\kappa r) + BJ_{-\frac{1}{3}}(\kappa r),$$

where

$$\kappa^2 = \tfrac{4}{9}W/lB \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12).$$

To satisfy the terminal conditions we have to take $A = 0$, and

$$J_{-\frac{1}{3}}(\kappa l^{\frac{3}{2}}) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13);$$

thus there will be no solution in the form proposed, or the straight form will be stable, provided $\kappa l^{\frac{3}{2}}$ is less than the least root of equation (13); this root is 1·88 nearly. Hence the critical length is approximately

$$2{\cdot}83\sqrt{(B/W)},$$

which is a little less than twice what it would be if the weight were concentrated at the highest point[1].

377. Stability of Twisted Rod.

Another very interesting problem of collapse is that which is presented by a twisted rod. Suppose a thin rod of uniform flexibility in all planes through its elastic central-line subjected to couples at its ends, and let the axes of the couples be parallel to the central-line of the rod in the unstrained state. Then one configuration of equilibrium is that in which the rod is straight and simply twisted. If τ be the twist the potential energy in this configuration is $\frac{1}{2}C\tau^2$, where C is the torsional rigidity; and if G be the applied couple $G = C\tau$, so that the potential energy is $\frac{1}{2}G^2/C$.

[1] The above solution is equivalent to that given by Prof. Greenhill in *Proc. Camb. Phil. Soc.* IV. 1881.

But there is another configuration in which the rod can be held by the terminal couples G, viz. we have seen in art. 239 that the rod can be held in the form of a helix of radius r and angle α, provided

$$G = B\cos\alpha/r,$$

where B is the flexural rigidity, and then the twist τ is given by the equation

$$G = C\sin\alpha\,\tau.$$

Now the potential energy in this configuration is

$$\tfrac{1}{2}\left(B\frac{\cos^4\alpha}{r^2} + C\tau^2\right),$$

or

$$\tfrac{1}{2}G^2\left(\frac{\cos^2\alpha}{B} + \frac{\sin^2\alpha}{C}\right),$$

and this is less or greater than the potential energy in the straight form according as

$$B > \text{or} < C.$$

For an isotropic rod of circular section (of radius c) $B = \frac{1}{4}E\pi c^4$, and $C = \frac{1}{2}\mu\pi c^4$, while $E = 2\mu(1+\sigma)$. Thus the straight form is unstable if the rod be entirely unsupported.

If however the ends of the rod are given points on a line parallel to the line of action of the couple, the only possible helical form will be one of angle very nearly $\frac{1}{2}\pi$ on a cylinder of very small radius, and the length of the rod must be an integral multiple of the length of a complete turn of the helix. Thus, if l be the length of the rod the least possible value of l is $2\pi r\sec\alpha$, or the critical length is

$$l = 2\pi B/G \qquad\qquad (14).$$

Thus we shew that if there is no force applied at the ends, but the ends are nevertheless fixed, the straight form will be unstable if the critical length given by (14) be exceeded.

The supposition of no terminal force appears extremely artificial and we shall proceed to consider the problem in a more general way.

378. Stability of Rod under Thrust and Couple.

Suppose that a naturally straight rod of uniform flexibility in all planes through its elastic central-line, has its ends supported at two given points, whose distance apart is equal to the natural length of the rod, and suppose that equal and opposite forces and

couples are applied at the ends, the axes of the couples coinciding with the line joining the fixed points; we shall seek the conditions that a configuration of equilibrium with displacements of pure bending and twisting[1] may be possible.

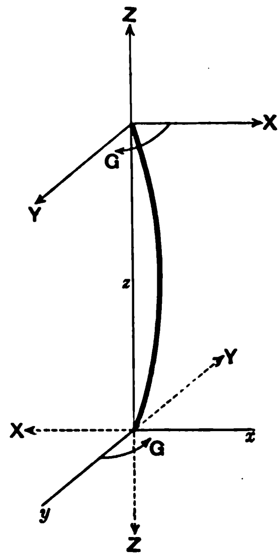


Fig. 58.

Let axes of x, y, z be drawn through one end, and let the forces and couple applied at the other end be X, Y, Z, G as shewn in the figure (fig. 58). Let x, y, z be the coordinates of any point on the elastic central-line in a configuration of equilibrium in which the displacements are very small, and ds the element of arc of the central-line. The stress-resultant at any section has components X, Y, Z (see art. 236).

Since the rod is of uniform flexibility in all planes through its elastic central-line, and is straight when unstrained, the twisting couple must be constant (p. 70). Let $C\tau$ be the twisting couple at any section. The flexural couples compound into a single couple B/ρ about the binormal to the elastic central-line, where ρ is the principal radius of curvature. Hence the components of the flexural couple at any section about the axes of x, y, z are

[1] *i.e.* of course the extension is of the second order of small quantities when the displacements are of the first order.

$$B\left(\frac{dy}{ds}\frac{d^2z}{ds^2}-\frac{dz}{ds}\frac{d^2y}{ds^2}\right),$$

and two similar expressions. The equations of equilibrium for the part between the origin and any point (x, y, z) are found by taking moments about the axes. Since x and y are very small while z is ultimately equal to s, their ultimate form is

$$\left.\begin{aligned} -B\frac{d^2y}{dz^2}+C\tau\frac{dx}{dz}+(yZ-zY)&=0,\\ B\frac{d^2x}{dz^2}+C\tau\frac{dy}{dz}+(zX-xZ)&=0,\\ C\tau\qquad +(xY-yX)&=G \end{aligned}\right\}\text{..........(15).}$$

The terminal conditions are that

$$x=y=0 \text{ when } z=0 \text{ and when } z=l,$$

and that the terminal stress-couples about the axes of x and y vanish. The latter conditions shew that, when $z=0$ or l

$$-B\frac{d^2y}{dz^2}+C\tau\frac{dx}{dz}=0,$$

$$B\frac{d^2x}{dz^2}+C\tau\frac{dy}{dz}=0.$$

To satisfy the last two conditions we have to take

$$X=Y=0,$$

so that the force at the ends must reduce to a tension (or thrust) along the axis z. The third of equations (15) then shews that $C\tau=G$.

It is now easy to prove, by solving the differential equations, that x and y must have the forms

$$x=A\sin mz\cos(nz+z_0),$$

$$y=A\sin mz\sin(nz+z_0),$$

where z_0 and A are arbitrary constants, and m and n are given by

$$m^2=\frac{G^2}{4B^2}-\frac{Z}{B},\qquad n=\frac{G}{2B};$$

and the conditions at the ends $z=l$ will be satisfied if

$$\frac{\pi^2}{l^2}=\frac{G^2}{4B^2}-\frac{Z}{B}\text{ .... (16).}$$

This equation consequently gives the least length l for which the straight form of the rod is unstable. It agrees with the result of art. 377 if $Z=0$.

If Z be negative, so that the rod is under thrust and couple, the result is

$$\frac{\pi^2}{l^2}=\frac{G^2}{4B^2}+\frac{R}{B},$$

where R is the thrust[1]. When we put $G=0$ this agrees with the result (3) of art. 373.

Taking Z positive, it will be impossible to satisfy the conditions at the end $z=l$ if $Z>\frac{1}{4}G^2/B$. Thus however great the twist τ may be it will always be possible to hold the rod straight by a tension exceeding $\frac{1}{4}C^2\tau^2/B$, provided this tension does not produce a finite extension of the rod.

379. Stability of Ring under External Pressure.

The only problem (so far as I am aware) concerning the stability of naturally curved rods which has been solved is the problem referred to in art. 231 of the collapse of a naturally circular ring under external pressure.

Let a be the radius of the ring in the unstrained state, B the flexural rigidity for bending in its plane, P the pressure per unit length, and ρ the radius of curvature after strain; then on p. 58 we have proved that in any mode of deformation involving no extension of the elastic central-line there is an equation of the form

$$\frac{B}{\rho}-\frac{B}{a}=\tfrac{1}{2}Pr^2+\text{const}.\qquad\qquad(17),$$

where r is the distance of any point on the strained elastic central-line from a fixed point in the plane.

Under very small pressures the ring will remain circular, and the radius will diminish under the pressure, but when the pressure exceeds a certain limit the ring will bend. When the critical pressure is just exceeded the deformation from the circular form

[1] This result was given by Prof. Greenhill in his article on the 'Strength of Shafting' in the *Proceedings of the Institute of Mechanical Engineers* for 1883. The problem of the stability of a shaft rotating between bearings is considered in the same article.

will be indefinitely small, and the above equation shews that r will be very nearly constant, so that the fixed point from which r is measured is the centre of the unstrained circle, and r is very nearly equal to a.

Let u be the displacement in the direction of the radius drawn inwards, and w the displacement in the direction of the tangent at any point of the central-line whose angular distance from a fixed point is θ. Then, since there is no extension we have, by (43) of art. 300,

$$\mathrm{u} = \frac{d\mathrm{w}}{d\theta}.$$

Also we have

$$r = a - \mathrm{u}, \qquad \frac{1}{\rho} - \frac{1}{a} = \frac{1}{a^2}\left(\frac{d^2\mathrm{u}}{d\theta^2} + \mathrm{u}\right).$$

Thus equation (17) becomes

$$\frac{B}{a^2}\left(\frac{d^2\mathrm{u}}{d\theta^2} + \mathrm{u}\right) = Pa\mathrm{u}.$$

Since u is periodic in θ we take $\mathrm{u} = \cos n(\theta + \alpha)$, where n is an integer, and we find

$$(n^2 - 1)B = Pa^3.$$

Hence, taking $n = 2$, we find the critical pressure given by the equation

$$P = 3B/a^3 \text{..........................(18)},$$

as stated in art. 231.

We could hence infer, by an appropriate change of constants, the condition of stability[1] of an infinitely long cylindrical shell under external pressure P per unit area in the form

$$P < 3C/a^3,$$

where C is the cylindrical rigidity $\frac{2}{3}Eh^3/(1 - \sigma^2)$ of art. 309.

We shall come upon this result again in connexion with the problem of the stability of boiler flues.

[1] This condition was first given by Mr Bryan, 'Application of the Energy Test to the Collapse of a long thin pipe under external pressure', *Proc. Camb. Phil. Soc.* vi. 1888. It has also been obtained by Mr Basset, 'On the Difficulties of constructing a Theory of the Collapse of Boiler-flues', *Phil. Mag.* 1892. The investigation given above is suggested by the work of M. Lévy (see art. 231). A new method will be given in art. 383 below.

380. Stability of Rectangular Plate under Thrust in its Plane.

We pass now to the consideration of some problems on the stability of thin plates and shells. The first of these that we shall take is the problem presented by a rectangular plate whose edges are supported and are subject to given thrusts. When the thrusts are not too great the plate simply contracts, and the middle-surface remains plane; and the displacements in such a configuration of equilibrium could be found from the equations of art. 327; but when the thrusts exceed certain limits the plate can buckle, and the conditions that this should be possible may be found by assuming that the departure of the strained middle-surface from the plane form is infinitesimal.

Suppose the thickness of the plate is $2h$, and the sides of the plate are of lengths a and b, and let us refer the plate to axes of x and y having the origin at one corner, so that the equations of the edges are $x = 0$, $y = 0$, $x = a$, $y = b$, and let w be the displacement of a point on the middle-surface in the direction of the normal to its plane. The changes of curvature are given (to a sufficient approximation) by

$$\kappa_1 = \frac{\partial^2 \mathrm{w}}{\partial x^2}, \qquad \kappa_2 = \frac{\partial^2 \mathrm{w}}{\partial y^2}, \qquad \tau = \frac{\partial^2 \mathrm{w}}{\partial x \partial y} \quad \text{.........(19).}$$

The flexural couples G_1, G_2 across the normal sections $x = \text{const.}$, and $y = \text{const.}$, and the torsional couple H are given (to a sufficient approximation) by

$$G_1 = -C(\kappa_1 + \sigma\kappa_2), \quad G_2 = C(\kappa_2 + \sigma\kappa_1), \quad H = C(1-\sigma)\tau \text{...(20),}$$

where C is the cylindrical rigidity $\frac{2}{3}Eh^3/(1-\sigma^2)$. The equations of equilibrium when forces are applied at the edges only are three equations of resolution and two equations of moments. The equations of resolution for finite displacements are (see p. 242)

$$\left.\begin{aligned} &\frac{\partial P_1}{\partial x} + \frac{\partial U_2}{\partial y} - T_1\kappa_1 - T_2\tau = 0, \\ &\frac{\partial U_1}{\partial x} + \frac{\partial P_2}{\partial y} - T_1\tau - T_2\kappa_2 = 0, \\ &\frac{\partial T_1}{\partial x} + \frac{\partial T_2}{\partial y} + P_1\kappa_1 + P_2\kappa_2 + (U_1 + U_2)\tau = 0 \end{aligned}\right\} \text{......(21),}$$

wherein P_1, U_1, T_1 are the stress-resultants parallel to x and y and normal to the strained middle-surface that act across the normal section $x = \text{const.}$, and U_2, P_2, T_2 are similar quantities for the normal section $y = \text{const.}$

The equations of moments are

$$\left.\begin{aligned}\frac{\partial H}{\partial x} + \frac{\partial G_2}{\partial y} + T_2 = 0,\\ \frac{\partial G_1}{\partial x} - \frac{\partial H}{\partial y} - T_1 = 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

The boundary-conditions at the edges are that $\mathrm{w} = 0$, and that when $x = 0$ or a

$$P_1 = -\mathfrak{P}_1, \quad U_1 + H/\rho_1' = 0, \quad G_1 = 0 \ldots\ldots\ldots (23);$$

also when $y = 0$ or b

$$P_2 = -\mathfrak{P}_2, \quad U_2 + H/\rho_2' = 0, \quad G_2 = 0 \ldots\ldots\ldots (24),$$

in which $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are the applied thrusts, and ρ_1', ρ_2' are the radii of curvature of the normal sections through the strained edge-lines.

Equations (22) shew that when w is small T_1 and T_2 are of the same order as w, and thus the terms such as $T_1\kappa_1$ can be omitted from equations (21), while the terms such as H/ρ_1' can be omitted from the boundary-conditions. The reduced equations can now be satisfied by supposing that U_1 and U_2 vanish, and P_1 and P_2 are constants; these constants are $-\mathfrak{P}_1$ and $-\mathfrak{P}_2$. The differential equation for w then becomes[1]

$$C\left[\frac{\partial^4 \mathrm{w}}{\partial x^4} + \frac{\partial^4 \mathrm{w}}{\partial y^4} + 2\frac{\partial^4 \mathrm{w}}{\partial x^2 \partial y^2}\right] + \mathfrak{P}_1 \frac{\partial^2 \mathrm{w}}{\partial x^2} + \mathfrak{P}_2 \frac{\partial^2 \mathrm{w}}{\partial y^2} = 0 \ldots\ldots (25).$$

We can find a solution of this equation which also satisfies the conditions at the boundaries in the form

$$\mathrm{w} = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

where m and n are integers, and the constant A is small but otherwise arbitrary, provided

$$C\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 = \left[\mathfrak{P}_1 \frac{m^2\pi^2}{a^2} + \mathfrak{P}_2 \frac{n^2\pi^2}{b^2}\right].$$

[1] This example shews very well the necessity for beginning with finite displacements. Equations such as (45) and (46) of art. 340 when applied to a naturally plane plate would not have led to equation (25).

The critical thrusts for which the plate just becomes unstable are therefore the least values of $\mathfrak{P}_1$ and $\mathfrak{P}_2$ for which such an equation as

$$C = \frac{1}{\pi^2} \frac{\mathfrak{P}_1 \dfrac{m^2}{a^2} + \mathfrak{P}_2 \dfrac{n^2}{b^2}}{\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2} \qquad \text{.....................(26)}$$

can be satisfied.

The following results[1] can be easily deduced:

(1°) When $\mathfrak{P}_1 = \mathfrak{P}_2 = \mathfrak{P}$ say, the least thrust for which the plate is unstable is

$$\mathfrak{P} = C\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2}\right).$$

(2°) When $\mathfrak{P}_2 = 0$ the least thrust will be found by putting $n = 1$, and m that integer for which the ratio a^2/b^2 lies between $m(m-1)$ and $m(m+1)$.

(3°) When $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are finite and different the plate will buckle into corrugations parallel to the side b (so that $n = 1, m \neq 1$) if $\mathfrak{P}_2 < \frac{1}{2}\mathfrak{P}_1$, and into corrugations parallel to the side a if $\mathfrak{P}_1 < \frac{1}{2}\mathfrak{P}_2$.

Note. When $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are equal the equation (25) becomes

$$C\nabla^4 w + \mathfrak{P}\nabla^2 w = 0,$$

the equation applied by Mr Bryan to the discussion of the stability of a clamped circular plate. (See the paper just quoted.) It is not difficult to shew that in this problem the least thrust $\mathfrak{P}$ consistent with instability is given by the equation

$$\mathfrak{P} = Cz^2/a^2,$$

where a is the radius of the plate, and z is the least root of the equation $J_1(z) = 0$.

381. Stability of infinite strip of plating under thrust.

We shall suppose that an infinitely long rectangular strip of breadth l is subject to uniform thrust at one edge, the opposite

[1] See Mr Bryan's paper 'On the Stability of a Plane Plate under Thrusts in its own Plane', *Proc. Lond. Math. Soc.* XXII. 1891, where the application to the stability of a structure supported on parallel ribs is given. A somewhat similar problem is presented by a plate to which bodily thrusts are applied. Thus a piece of card on which a photograph has been pasted tends to wrinkle as the surface dries.

edge being supported or built-in. If the thrust be sufficiently great the middle-surface will become a cylinder with generators parallel to the loaded edge, and the case therefore falls under the second class of cases noticed on p. 242 for which the equations of finite deformation can be written down.

If we take axes of α and β on the unstrained middle-surface, of which the axis β is parallel to the loaded edge, α and β will be parameters defining a point on the strained middle-surface. Suppose ρ is the finite principal radius of curvature of the strained middle-surface[1]; in the notation of art. 341 $-1/\rho$ is to be written for κ_1, and κ_2 and τ are both zero. All the quantities that occur are independent of β, and the equations of equilibrium become

$$\left.\begin{array}{lll} \dfrac{\partial P_1}{\partial \alpha}+\dfrac{T_1}{\rho}=0, & \dfrac{\partial U_1}{\partial \alpha}=0, & \dfrac{\partial T_1}{\partial \alpha}-\dfrac{P_1}{\rho}=0, \\ T_2=0, & \dfrac{\partial G_1}{\partial \alpha}-T_1=0 & \end{array}\right\} \text{......} (27).$$

The couples will be

$$G_1=C/\rho,\quad G_2=-C\sigma/\rho,\quad H=0 \text{............} (28).$$

The boundary-conditions at the loaded edge are that P_1 and T_1 have values depending on the applied thrust and the inclination of the strained to the unstrained middle-surface, while G_1 and U_1 vanish.

The first, third and fifth of equations (27) are alone significant, and by appropriate changes of notation they may be identified with equations (1) of art. 214 by which the flexure of a thin rod is determined.

It follows that the problem of the stability of the infinite strip under edge-thrust is identical in form with the problem of the stability of a thin rod under a terminal load parallel to its unstrained elastic central-line[2]. This is the problem considered in art. 373.

[1] The sign of ρ has been changed because in treating plates and shells we have estimated the curvatures positive outwards, and in the theory of wires with which we wish to effect a comparison the curvature is estimated positive inwards.

[2] It is also noteworthy that when the edge-thrust exceeds the limit for which the plane form becomes unstable the strip bends into a cylinder whose normal section is an *elastica*.

We conclude[1] that, if the opposite edge be simply supported, the strip becomes unstable when the thrust $\mathfrak{P}$ exceeds the limit $C\pi^2/l^2$, and if the opposite edge be built-in the limit is $\frac{1}{4}C\pi^2/l^2$.

382. Stability and strength of boilers and boiler-flues.

We shall conclude this chapter with an account of the stability and strength of boilers and boiler-flues. A boiler consists essentially of a thin-walled hollow cylinder containing hot water and steam at high pressure; the water is heated by the passage of hot gases from the furnace along thin-walled cylindrical flues which run from end to end of the boiler, and the length of the boiler is generally maintained nearly constant by a more or less elaborate system of stays. It is found that the flues tend to collapse under the external pressure of the steam, and to avoid this tendency a flue is frequently made in detached pieces connected by massive flanged joints, or some other device is adopted for the purpose of shortening the effective length. We have now to explain why a long flue tends to collapse under external pressure and a short flue can resist this tendency.

Consider the problem of finding the displacement in a short cylinder with plane ends, whose surface is subjected to uniform hydrostatic pressure. From the fact that a closed surface cannot be bent without stretching it follows at once that finite displacements are geometrically impossible, and the nearly cylindrical form cannot be unstable.

If, however, the ends of the cylinder be so distant that their effect may be disregarded the cylinder may be treated as infinite, and then we have already seen (art. 379) that there is a critical pressure which cannot be exceeded without instability.

We shall now give a direct investigation of the critical pressure under which the infinite cylinder becomes unstable, and we shall then investigate the standard length in comparison with which the length of the cylinder must be great in order that it may be treated as infinite.

383. Infinite cylindrical shell under uniform external pressure.

When the external pressure is small the shell contracts radially, and the expression for the radial displacement has been given in

[1] This result is otherwise obtained by Mr Bryan in his paper 'On the Stability of Elastic Systems', *Proc. Camb. Phil. Soc.* VI. 1888.

I., art. 130. To find the condition of instability of this configuration of equilibrium we have to suppose that the section is held under the external pressure in a shape slightly displaced from the circular form by pure bending without stretching of the middle-surface.

Suppose the unstrained middle-surface of the shell is of radius a, and (as in art. 350) take x for the length measured along a generator from a fixed normal section, and $a\phi$ for the length measured along the section from a fixed generator, let u be the displacement along the generator, v the displacement along the tangent to the circular section, and w the displacement along the normal to the cylinder drawn outwards. According to art. 352, the only displacements of pure bending which remain everywhere small are independent of x and satisfy the equations

$$\mathrm{u}=0, \qquad \frac{\partial \mathrm{v}}{\partial \phi}+\mathrm{w}=0 \text{ .... } (29),$$

so that the middle-surface is always a cylindrical surface, but the normal section does not remain circular.

Let P_1, U_1, T_1 be the stress-resultants per unit length of a normal section, across such a section of the strained middle-surface, parallel respectively to the generator, the tangent to the section, and the normal to the surface, and let U_2, P_2, T_2 be the stress-resultants per unit length of a generator, across a section through that generator of the strained middle-surface, in the same directions. Also let G_1, G_2 be the flexural couples across the same two sections, and H the torsional couple. Since the cylinder is strained into a cylinder with the same generators the problem falls under the first class of cases mentioned in art. 341 for which the equations of equilibrium with finite displacements can be written down, and these equations are simplified by the fact that the principal curvatures in normal sections through the generators are zero both before and after strain. The flexural couples are given by the equations

$$G_1=C\sigma\left(\frac{1}{\rho}-\frac{1}{a}\right), \qquad G_2=-C\left(\frac{1}{\rho}-\frac{1}{a}\right), \qquad H=0 \ldots (30),$$

where ρ is the radius of curvature of the strained section estimated positive inwards. In the equations of equilibrium we have to put

$$A=1, \quad B=a, \quad \alpha=x, \quad \beta=\phi, \quad 1/\rho_1'=0, \quad 1/\rho_2'=-1/\rho \ldots (31);$$

so that the equations of resolution become

$$\left.\begin{aligned}\frac{\partial P_1}{\partial x}+\frac{1}{a}\frac{\partial U_2}{\partial \phi}=0,\\ \frac{\partial U_1}{\partial x}+\frac{1}{a}\frac{\partial P_2}{\partial \phi}+\frac{T_2}{\rho}=0,\\ \frac{\partial T_1}{\partial x}+\frac{1}{a}\frac{\partial T_2}{\partial \phi}-\frac{P_2}{\rho}-\Pi=0\end{aligned}\right\}\quad\text{.................(32)},$$

where Π is the resultant of the internal and external pressures per unit area estimated positive inwards. The equations of moments are

$$\frac{1}{a}\frac{\partial G_2}{\partial \phi}+T_2=0,\qquad \frac{\partial G_1}{\partial x}-T_1=0\text{ ....(33).}$$

We can find a solution independent of x by taking the second and third of equations (32), and the first of equations (33), and, on eliminating T_2, these equations give us

$$\frac{\partial P_2}{\partial \phi}-\frac{1}{\rho}\frac{\partial G_2}{\partial \phi}=0,\qquad \frac{\rho}{a^2}\frac{\partial^2 G_2}{\partial \phi^2}+P_2=-\Pi\rho;$$

from these, eliminating P_2, we have

$$\frac{\partial}{\partial \phi}\left(\frac{\rho}{a^2}\frac{\partial^2 G_2}{\partial \phi^2}\right)+\frac{1}{\rho}\frac{\partial G_2}{\partial \phi}=-\Pi\frac{\partial \rho}{\partial \phi}\text{ ....(34).}$$

When the displacement from the circular form is infinitesimal

$$\frac{1}{\rho}=\frac{1}{a}-\frac{1}{a^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \phi^2}+\mathrm{w}\right),\qquad G_2=\frac{C}{a^2}\left(\frac{\partial^2 \mathrm{w}}{\partial \phi^2}+\mathrm{w}\right),$$

and, on rejecting terms of the second order in w, we find

$$\frac{C}{a^3}\left(\frac{\partial^2}{\partial \phi^2}+1\right)^2\frac{\partial \mathrm{w}}{\partial \phi}=-\Pi\left(\frac{\partial^2}{\partial \phi^2}+1\right)\frac{\partial \mathrm{w}}{\partial \phi}\text{ ....(35).}$$

We may assume a solution of this equation in the form

$$\partial \mathrm{w}/\partial \phi = A\cos(n\phi+\phi_0),$$

provided $\Pi=(n^2-1)\,C/a^3$, and, since $\partial \mathrm{w}/\partial \phi$ must be periodic in ϕ with a period 2π, n must be an integer.

Hence the least value of Π consistent with instability is

$$\Pi=3C/a^3=\frac{2E}{1-\sigma^2}\frac{h^3}{a^3}\text{ .... (36),}$$

as already otherwise proved in art. 379.

When Π has this value[1] the cylinder can suffer deformations of pure bending, and the displacements have the forms

$$\mathrm{u} = 0, \quad \mathrm{v} = \tfrac{1}{2}A\cos 2\phi, \quad \mathrm{w} = A\sin 2\phi,$$

where A is a small arbitrary constant. The deformation is maintained by the pressure Π.

When the flue is of finite length the conditions at the ends make this solution fail; but, if the ends be sufficiently distant from the middle to be disregarded, this solution ought to hold near the middle of the flue.

384. Effect of ends of flue.

We have now to shew how the effect of the ends falls off in a very long flue, and for this purpose we shall consider the flue to be terminated by a rigid plane end at $x = 0$, and to extend to $x = \infty$ in the positive direction. For all very great values of x we shall suppose that the displacement is one of pure bending given by the equations

$$\mathrm{u} = 0, \quad \mathrm{v} = \tfrac{1}{2}A\cos 2\phi, \quad \mathrm{w} = A\sin 2\phi \quad \text{.........(37)},$$

and we shall investigate displacements which satisfy the equations of equilibrium and these conditions at $x = \infty$, and also satisfy the further conditions that v and w vanish with x.

Now by means of displacements of the forms given above we can satisfy the equations when Π is retained. If therefore we take

$$\mathrm{u} = \mathrm{u}', \quad \mathrm{v} = \mathrm{v}' + \tfrac{1}{2}A\cos 2\phi, \quad \mathrm{w} = \mathrm{w}' + A\sin 2\phi,$$

u′, v′, w′ must be displacements which satisfy the equations of equilibrium under no forces, and also satisfy the boundary-conditions that u′, v′, w′ vanish at $x = \infty$, and that v, w vanish at $x = 0$.

It is clear at once that the equations and conditions will not admit of a solution if they be limited by supposing either that the displacements u′, v′, w′ are non-extensional, or that the stretching is the important thing; for all the forms of displacement by pure

[1] Taking a steel boiler flue of 36 in. diameter and $\frac{3}{8}$ in. thickness, and putting $E = 2{\cdot}45 \times 10^{12}$ dynes per sq. cm., and $\sigma = \frac{1}{4}$, we find $\Pi = 81$ lbs. wt. per sq. in. nearly.

bending possible in a cylinder have been obtained in art. 352, and none of them satisfy all the conditions; on the other hand there are no extensional displacements which can satisfy the boundary-conditions. The case therefore falls into the exceptional third class of art. 321, in which terms depending on extension and terms depending on change of curvature must both be retained. This complicates the problem; but, on the other hand, as we have already noticed in art. 336, in the third class of cases the second approximation to the stress-resultants P_1, P_2, U_1, U_2 becomes unimportant, and we therefore have by art. 339 the following expressions for the stress-resultants:—

$$P_1 = \frac{3C}{h^2}(\epsilon_1 + \sigma\epsilon_2), \quad P_2 = \frac{3C}{h^2}(\epsilon_2 + \sigma\epsilon_1), \quad U_1 = U_2 = \frac{3C}{2h^2}(1-\sigma)\varpi \qquad \text{.............(38)},$$

where h is the half-thickness of the cylindrical shell, C the cylindrical rigidity, ϵ_1 the extension along the generator, ϵ_2 the extension along the circular section, and ϖ the shear of these two lines. The stress-couples are given by the equations

$$G_1 = -C(\kappa_1 + \sigma\kappa_2), \quad G_2 = C(\kappa_2 + \sigma\kappa_1), \quad H = C(1-\sigma)\tau \ldots(39),$$

in which κ_1, κ_2, τ are the 'changes of curvature'. Further, we have seen already in art. 350, that in the case of a cylindrical shell of radius a

$$\left.\begin{array}{lll} \epsilon_1 = \dfrac{\partial u'}{\partial x}, & \epsilon_2 = \dfrac{1}{a}\left(\dfrac{\partial v'}{\partial \phi} + w'\right), & \varpi = \dfrac{\partial v'}{\partial x} + \dfrac{1}{a}\dfrac{\partial u'}{\partial \phi}, \\ \kappa_1 = \dfrac{\partial^2 w'}{\partial x^2}, & \kappa_2 = \dfrac{1}{a^2}\left(\dfrac{\partial^2 w'}{\partial \phi^2} - \dfrac{\partial v'}{\partial \phi}\right), & \tau = \dfrac{1}{a}\left(\dfrac{\partial^2 w'}{\partial x \partial \phi} - \dfrac{\partial v'}{\partial x}\right) \end{array}\right\}\ldots(40).$$

We can therefore express the stress-resultants and stress-couples in terms of the displacements.

The equations of equilibrium under no forces are

$$\left.\begin{array}{l} \dfrac{\partial P_1}{\partial x} + \dfrac{1}{a}\dfrac{\partial U_2}{\partial \phi} = 0, \quad \dfrac{\partial U_1}{\partial x} + \dfrac{1}{a}\dfrac{\partial P_2}{\partial \phi} + \dfrac{T_2}{a} = 0, \quad \dfrac{\partial T_1}{\partial x} + \dfrac{1}{a}\dfrac{\partial T_2}{\partial \phi} - \dfrac{P_2}{a} = 0, \\ \dfrac{\partial H}{\partial x} + \dfrac{1}{a}\dfrac{\partial G_2}{\partial \phi} + T_2 = 0, \quad \dfrac{\partial G_1}{\partial x} - \dfrac{1}{a}\dfrac{\partial H}{\partial \phi} - T_1 = 0 \end{array}\right\} \qquad \text{....................(41)}.$$

If we eliminate T_1 and T_2 from the first three of these by using the fourth and fifth, and substitute for the remaining stress-couples and stress-resultants their values in terms of displacements, we find the three following equations:

$$\frac{\partial}{\partial x}\left[\frac{\partial u'}{\partial x}+\frac{\sigma}{a}\left(\frac{\partial v'}{\partial\phi}+w\right)\right]+\frac{1-\sigma}{2a}\frac{\partial}{\partial\phi}\left(\frac{\partial v'}{\partial x}+\frac{1}{a}\frac{\partial u'}{\partial\phi}\right)=0,$$

$$\frac{(1-\sigma)}{2}\frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial x}+\frac{1}{a}\frac{\partial u'}{\partial\phi}\right)+\frac{1}{a}\frac{\partial}{\partial\phi}\left[\sigma\frac{\partial u'}{\partial x}+\frac{1}{a}\left(\frac{\partial v'}{\partial\phi}+w'\right)\right]$$

$$-\tfrac{1}{3}h^2\left[\frac{1-\sigma}{a^2}\left(\frac{\partial^3 w'}{\partial x^2\partial\phi}-\frac{\partial^2 v'}{\partial x^2}\right)+\frac{1}{a^2}\frac{\partial}{\partial\phi}\left\{\frac{1}{a^2}\left(\frac{\partial^2 w'}{\partial\phi^2}-\frac{\partial v'}{\partial\phi}\right)+\sigma\frac{\partial^2 w'}{\partial x^2}\right\}\right]=0,$$

and

$$+\frac{\sigma}{a}\frac{\partial u'}{\partial x}+\frac{1}{a^2}\left(w'+\frac{\partial v'}{\partial\phi}\right)$$

$$+\tfrac{1}{3}h^2\left[\frac{\partial^4 w'}{\partial x^4}+\frac{\sigma}{a^2}\left(\frac{\partial^4 w'}{\partial x^2\partial\phi^2}-\frac{\partial^3 v'}{\partial x^2\partial\phi}\right)+\frac{1-\sigma}{a^2}\left(\frac{\partial^4 w'}{\partial x^2\partial\phi^2}-\frac{\partial^3 v'}{\partial x^2\partial\phi}\right)\right]$$

$$+\tfrac{1}{3}h^2\left[\frac{1-\sigma}{a^2}\left(\frac{\partial^4 w'}{\partial x^2\partial\phi^2}-\frac{\partial^3 v'}{\partial x^2\partial\phi}\right)+\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}\left\{\frac{1}{a^2}\left(\frac{\partial^2 w'}{\partial\phi^2}-\frac{\partial v'}{\partial\phi}\right)+\sigma\frac{\partial^2 w'}{\partial x^2}\right\}\right]=0.$$

Now with a view to the boundary-conditions at $x=\infty$ we shall take

$$u'\propto\sin 2\phi,\quad v'\propto\cos 2\phi,\quad w'\propto\sin 2\phi.$$

Then the above equations will become simultaneous linear equations to determine u', v', w' as functions of x, and to solve them we must suppose that u', v', w' involve x by containing a factor of the form $e^{\lambda x}$; we thus find the following equation for λ:

$$\begin{vmatrix} \lambda^2-\frac{2}{a^2}(1-\sigma) & -\frac{(1+\sigma)\lambda}{a} & \frac{\sigma\lambda}{a} \\ \frac{(1+\sigma)\lambda}{a} & \frac{1-\sigma}{2}\lambda^2-\frac{4}{a^2}+\frac{h^2}{3}\left(\lambda^2\frac{1-\sigma}{a^2}-\frac{4}{a^4}\right) & \frac{2}{a^2}-\frac{h^2}{3}\left(\frac{2\lambda^2}{a^2}-\frac{8}{a^4}\right) \\ \frac{\sigma\lambda}{a} & -\frac{2}{a^2}+\frac{h^2}{3}\left(2\lambda^2\frac{2-\sigma}{a^2}-\frac{8}{a^4}\right) & \frac{1}{a^2}+\frac{h^2}{3}\left(\lambda^4-\frac{8\lambda^2}{a^2}+\frac{16}{a^4}\right) \end{vmatrix}=0 \qquad \text{.................................(42).}$$

We have to approximate to the values of λ when h is small. Since the terms of highest degree contain h^2 as a factor it is clear that some roots tend to become infinite as h is diminished indefinitely. These roots will be found by supposing that $h^2\lambda^4$ is finite and of the order a^{-2}. Retaining only the most important terms our equation becomes

$$\frac{1-\sigma}{2}\lambda^4\left[\frac{1}{a^2}+\frac{h^2}{3}\lambda^4-\frac{\sigma^2}{a^2}\right]=0,$$

from which
$$\lambda^4=-3\frac{1-\sigma^2}{h^2a^2}.$$

If then we write[1]

$$\lambda^2=2m^2\iota,\quad m=\frac{[\frac{3}{4}(1-\sigma^2)]^{\frac{1}{4}}}{\sqrt{(ha)}},$$

we shall have solutions in the form

$$\left.\begin{aligned}u'&=e^{-mx}(A_1\cos mx+B_1\sin mx)\sin 2\phi,\\ v'&=e^{-mx}(A_2\cos mx+B_2\sin mx)\cos 2\phi,\\ w'&=e^{-mx}(A_3\cos mx+B_3\sin mx)\sin 2\phi\end{aligned}\right\}\quad\ldots\ldots\ldots(43),$$

where m is large of the order of the reciprocal of the mean proportional between the thickness and the diameter.

The constants A_1, A_2, ... are not independent, and to find the relations among them we may take any two of the differential equations for u′, v′, w′. For this purpose it is convenient to rewrite the solutions in the form

$$u'=A_1'e^{\lambda x}\sin 2\phi,\quad v'=A_2'e^{\lambda x}\cos 2\phi,\quad w'=A_3'e^{\lambda x}\sin 2\phi,$$

where the coefficients are complex, and use the second and third of the differential equations to eliminate A_1'. We thus find (retaining only the most important terms)

$$\sigma\frac{1-\sigma}{2}\lambda^2a^2A_2'+\left\{2\sigma-(1+\sigma)\left(1+\frac{a^2h^2\lambda^4}{3}\right)\right\}A_3'=0,$$

or
$$A_3'=-\frac{m^2a^2}{2+\sigma}\iota A_2'.$$

Hence, if $\quad v'=e^{-mx}(A_2\cos mx+B_2\sin mx)\cos 2\phi$

be the form for v′, the corresponding form for w′ may be taken to be

$$w'=\frac{m^2a^2}{2+\sigma}e^{-mx}(B_2\cos mx-A_2\sin mx)\sin 2\phi.$$

To the solutions here obtained we add the solutions

$$u=0,\quad v=\tfrac{1}{2}A\cos 2\phi,\quad w=A\sin 2\phi,$$

[1] The quantity m when $\sigma=\frac{1}{4}$ is about 1·83 of the reciprocal of the mean proportional between the thickness and the diameter, or the reciprocal of m is about ·546 of this mean proportional.

which, as we have seen in the previous article, satisfy the differential equations of equilibrium (with Π retained) when the displacements tend to become finite, we then have to satisfy the boundary-conditions $\mathrm{v}=0$ and $\mathrm{w}=0$ at $x=0$, and we thus find

$$\left.\begin{array}{l}\frac{1}{2}A+A_2=0,\\ A+m^2a^2B_2/(2+\sigma)=0\end{array}\right\}\text{..................}(44).$$

We conclude that it is possible to add to the non-extensional solution a solution involving both bending and stretching by which the boundary-conditions can be satisfied. On inspecting the results we see that at any distance x from the end which is great compared with a mean proportional between the thickness and the diameter of the shell the influence of the ends becomes negligible.

If we apply the result obtained to the numerical example of a flue of $\frac{3}{8}$ in. thickness and 3 ft. diameter we find the mean proportional equal to three inches nearly. The influence of the ends will be practically negligible when the length exceeds some ten or twelve times this, and it will be necessary to shorten the effective length of the flue to somewhere about 3 ft. in order that the ends may sensibly stiffen it against collapse. This might be effected by means of rings placed at distances of about 3 ft. along the length of the flue, or by making the flue in detached pieces of about this length with massive flanged joints as is customary in stationary boiler practice.

Combining the results of this and the previous article we conclude—

(1) that no flue however long can collapse unless the pressure exceed

$$\frac{2E}{1-\sigma^2}\frac{h^3}{a^3};$$

(2) that when the pressure exceeds this limit any flue will collapse if its length exceed a certain multiple of the mean proportional between the diameter and the thickness.

The most important practical conclusion regards the spacing of the joints by which the flue is protected against collapse. We have the rule—the ratio of the distance between consecutive joints to the mean proportional between the thickness and the diameter must not exceed a certain limit. The limit should be determined by experiment.

The practical rule adopted by engineers[1] is to make the distance between the joints a certain multiple of the thickness, (frequently about 100 for a pressure of 90 lbs. wt. on the square inch,) and it is customary to diminish the ratio of these quantities when the working pressure is increased. The theory here given suggests nothing in regard to the change to be made in the spacing of the joints for a given change of pressure, since it is worked out on the supposition that the pressure is the least critical pressure, but it suggests a modification of the rule for spacing the joints when the pressure is given.

385. Strength of cylindrical shell.

In further illustration of the theory of thin shells under pressure we shall proceed to consider the symmetrical displacements produced in a thin cylindrical shell by uniform internal pressure. This will lead to an estimate of the strength of the outer shells of boilers.

Suppose Π is the excess of the internal pressure above the external. With the notation previously employed the displacement v vanishes, and all the displacements and stresses are independent of ϕ, and the equations of equilibrium become

$$\frac{\partial P_1}{\partial x}=0, \qquad \frac{\partial^2 G_1}{\partial x^2}-\frac{P_2}{a}=-\Pi \quad \text{..............(45).}$$

As in the last article we cannot take the displacement purely extensional for we should then be unable to satisfy the boundary-conditions, nor can we regard it as consisting of pure bending; we have, as before, to take for P_1 and P_2 their expressions in terms of the extension of the middle-surface, rejecting the second approximations of art. 339. In the above equations we have to put

$$P_1=\frac{3C}{h^2}(\epsilon_1+\sigma\epsilon_2), \qquad P_2=\frac{3C}{h^2}(\epsilon_2+\sigma\epsilon_1), \qquad G_1=-C\kappa_1,$$

where
$$\epsilon_1=\frac{\partial \mathrm{u}}{\partial x}, \qquad \epsilon_2=\frac{\mathrm{w}}{a}, \qquad \kappa_1=\frac{\partial^2 \mathrm{w}}{\partial x^2}.$$

The equations of equilibrium thus become

$$\left.\begin{aligned} &\frac{\partial \mathrm{u}}{\partial x}+\sigma\frac{\mathrm{w}}{a}=\frac{h^2P_1}{3C}, \text{ some constant,}\\ &C\left[\frac{\partial^4 \mathrm{w}}{\partial x^4}+\frac{3}{h^2}\left(\frac{\mathrm{w}}{a^2}+\frac{\sigma}{a}\frac{\partial \mathrm{u}}{\partial x}\right)\right]=\Pi \end{aligned}\right\}\text{............(46).}$$

[1] I am indebted for information on this subject to Mr W. H. Maw, Editor of *Engineering*.

Eliminating u we obtain the equation for w

$$\frac{\partial^4 \mathrm{w}}{\partial x^4} + 3\frac{1-\sigma^2}{h^2a^2}\mathrm{w} = \frac{\Pi a - \sigma P_1}{Ca}.$$

Taking the case of a cylinder of length $2l$ with plane rigid ends whose distance apart is maintained constant, and supposing that the boundary-conditions are that u, w, and $\partial \mathrm{w}/\partial x$ vanish when $x = \pm l$, and writing as in the last article $m^4 = \frac{3}{4}(1-\sigma^2)/h^2a^2$, it is not difficult to shew[1] that the solution of the equations (46) takes the form

$$\left.\begin{aligned} \mathrm{w} &= \frac{\Pi a - \sigma P_1}{4Cam^4}[1 - 2f(x, l)], \\ \mathrm{u} &= \frac{1}{4Ca^2m^5}[2\sigma(\Pi a - \sigma P_1)\phi(x, l) - (\sigma\Pi a - P_1)mx], \end{aligned}\right\}$$

in which $f(x, l)$ and $\phi(x, l)$ stand respectively for the functions

$$\frac{(\sinh ml \cos ml + \cosh ml \sin ml)\cosh mx \cos mx - (\sinh ml \cos ml - \cosh ml \sin ml)\sinh mx \sin mx}{\sinh 2ml + \sin 2ml},$$

and

$$\frac{\sinh ml \cos ml \sinh mx \cos mx + \cosh ml \sin ml \cosh mx \sin mx}{\sinh 2ml + \sin 2ml}.$$

Further, the stress-resultant P_1 is given by the equation

$$(\sigma\Pi a - P_1)ml = \sigma(\Pi a - \sigma P_1)\frac{\cosh 2ml - \cos 2ml}{\sinh 2ml + \sin 2ml}.$$

To estimate the strength of the shell it would be necessary to find the greatest extension. The work would be very complicated, but it is not difficult to see that what is required is the maximum of $\frac{\partial \mathrm{u}}{\partial x} \pm h\frac{\partial^2 \mathrm{w}}{\partial x^2}$. If we take the upper sign the value of this quantity when $x = l$ is the extension of the innermost longitudinal fibres parallel to the generators, and if ml be supposed indefinitely great the approximate value of this is

$$\frac{h^2}{3C}\Pi a[\sigma + \sqrt{\{3(1-\sigma^2)\}}].$$

[1] The results only are stated and the analysis left to the reader.

Taking $\sigma = \frac{1}{4}$, and remembering the value of C, we find that this strain is approximately equal to $\cdot 9033\ \Pi a/Eh$. This is not the greatest strain in the material, but if we allow a factor of safety[1] Φ we may take the condition of safety to be

$$\cdot 9033\ \Pi a/h < T_0/\Phi,$$

where T_0 is the breaking stress of the material found by pure traction experiments. For a steel boiler such as is used in locomotive engines, under an internal pressure of 180 lbs. wt. on the sq. in., with a diameter of 4 ft., and plates of thickness $\frac{13}{32}$ in., using the value $T_0 = 7\cdot 93 \times 10^9$ dynes per sq. cm., we find $\Phi = 6\cdot 4$. We may infer that the condition of safety of a cylindrical boiler with plane ends is of the form

$$\frac{h}{a} > n\frac{\Pi}{T_0},$$

where n is a number which may be taken equal to or a little less than 6 when the material is mild steel.

[1] The term 'factor of safety' is here used in a sense different to that in Vol. I. There $T_0/E\Phi$ is the greatest strain consistent with safety, here it is the greatest easily calculable strain consistent with safety.

NOTES.

Note A. On the stress-couples in a wire naturally curved.

It is proper to mention that the formulæ (38) of art. 298 for the couples G_1, G_2 are not in agreement with those obtained by Mr Basset in his paper 'On the Theory of Elastic Wires' (*Proc. Lond. Math. Soc.* XXIII. 1892). Mr Basset's values would be found by omitting the term $-\frac{1}{\sigma}\left(\frac{\partial \mathrm{u}}{\partial s}-\frac{\mathrm{v}}{\sigma}+\frac{\mathrm{w}}{\rho}\right)$ in the expression for G_1, and the term $-\frac{1}{\sigma}\left(\frac{\partial \mathrm{v}}{\partial s}+\frac{\mathrm{u}}{\sigma}\right)$ in the expression for G_2, and adding in the expression for G_2 a term depending on the extensions of line-elements initially coinciding with the tangent and principal normal. If the values thus obtained were the correct values then the stress-couples would not be proportional to the changes in the components of curvature as proved originally by Clebsch.

Note B. On the formulæ of art. 316.

We give here the proofs of certain formulæ due to M. Codazzi relating to the curvature of surfaces. The system of parameters α, β and the moving system of axes of coordinates x, y, z have been described in art. 316.

The line-element ds is given by the equation

$$ds^2=A^2d\alpha^2+B^2d\beta^2+2ABd\alpha d\beta\cos\chi \qquad (1).$$

The axis x at any point (α, β) is along the line $\beta=$const. whose element is $Ad\alpha$, the axis y is in the tangent plane and perpendicular to the axis x, and the axis z is the normal to the surface. The component translations of the origin when we pass from the point (α, β) to the point $(\alpha+d\alpha, \beta+d\beta)$ are $Ad\alpha+Bd\beta\cos\chi$, $Bd\beta\sin\chi$. The component rotations executed by the axes about themselves when this change of origin is made are $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$, where

$$\left.\begin{aligned}\delta\theta_1&=p_1d\alpha+p_2d\beta,\\ \delta\theta_2&=q_1d\alpha+q_2d\beta,\\ \delta\theta_3&=r_1d\alpha+r_2d\beta\end{aligned}\right\} \qquad (2).$$

Let λ, μ, ν be the direction-cosines of a line drawn through (α, β) referred to the axes of x, y, z at (α, β), and let l, m, n be the direction-cosines of the same line referred to axes fixed in space. Also let $\lambda+d\lambda$, $\mu+d\mu$, $\nu+d\nu$ be the direction-cosines of a neighbouring line through $(\alpha+d\alpha, \beta+d\beta)$ referred to the axes of x, y, z at $(\alpha+d\alpha, \beta+d\beta)$, and let $l+dl$, $m+dm$, $n+dn$ be the direction-cosines of this second line referred to the fixed axes. Then we may take the fixed axes to coincide with the axes of x, y, z at (α, β), and when this is done l, m, n are identical with λ, μ, ν, but dl, dm, dn are not identical with $d\lambda, d\mu, d\nu$. In fact we have

$$\left.\begin{aligned} dl &= d\lambda - \mu\delta\theta_3 + \nu\delta\theta_2, \\ dm &= d\mu - \nu\delta\theta_1 + \lambda\delta\theta_3, \\ dn &= d\nu - \lambda\delta\theta_2 + \mu\delta\theta_1 \end{aligned}\right\} \qquad (3).$$

In these formulæ any differential as dl means $(\partial l/\partial\alpha)\, d\alpha + (\partial l/\partial\beta)\, d\beta$.

Substituting for $\delta\theta_1$, $\delta\theta_2$, $\delta\theta_3$ from (2), and equating coefficients of $d\alpha$ and $d\beta$, we find such formulæ as

$$\frac{\partial l}{\partial\alpha} = \frac{\partial\lambda}{\partial\alpha} - \mu r_1 + \nu q_1.$$

Important formulæ are derived by taking the case in which the line (λ, μ, ν) is parallel to a line fixed in space. In that case dl, dm, dn vanish, and we find the six formulæ:

$$\left.\begin{aligned} \frac{\partial\lambda}{\partial\alpha} &= \mu r_1 - \nu q_1, & \frac{\partial\lambda}{\partial\beta} &= \mu r_2 - \nu q_2, \\ \frac{\partial\mu}{\partial\alpha} &= \nu p_1 - \lambda r_1, & \frac{\partial\mu}{\partial\beta} &= \nu p_2 - \lambda r_2, \\ \frac{\partial\nu}{\partial\alpha} &= \lambda q_1 - \mu p_1, & \frac{\partial\nu}{\partial\beta} &= \lambda q_2 - \mu p_2 \end{aligned}\right\} \qquad (4).$$

These are really formulæ of differentiation for λ, μ, ν. We may apply them to differentiate any identity in which λ, μ, ν occur, and in particular we may differentiate the formulæ (4). Taking the first two we find

$$\frac{\partial^2\lambda}{\partial\alpha\partial\beta} = \mu\frac{\partial r_1}{\partial\beta} - \nu\frac{\partial q_1}{\partial\beta} + r_1(\nu p_2 - \lambda r_2) - q_1(\lambda q_2 - \mu p_2),$$

$$\frac{\partial^2\lambda}{\partial\alpha\partial\beta} = \mu\frac{\partial r_2}{\partial\alpha} - \nu\frac{\partial q_2}{\partial\alpha} + r_2(\nu p_1 - \lambda r_1) - q_2(\lambda q_1 - \mu p_1).$$

By equating these values of $\partial^2\lambda/\partial\alpha\partial\beta$ we obtain a linear relation between μ and ν which holds for all values of the ratio $\mu : \nu$. We may therefore equate coefficients of μ and ν and obtain the equations

$$\left.\begin{aligned} \frac{\partial q_1}{\partial\beta} - \frac{\partial q_2}{\partial\alpha} &= r_1 p_2 - r_2 p_1, \\ \frac{\partial r_1}{\partial\beta} - \frac{\partial r_2}{\partial\alpha} &= p_1 q_2 - p_2 q_1 \end{aligned}\right\} \qquad (5).$$

These are two of the equations (6) on p. 203, and the third like formula can be found in the same way by comparing two values for $\partial^2\mu/\partial\alpha\partial\beta$.

Now consider any point Q and a neighbouring point Q'. Let x, y, z be the coordinates of Q referred to the moving axes at (α, β), and ξ, η, ζ the coordinates of the same point referred to the fixed axes. Also let $x+dx$, $y+dy$, $z+dz$ be the coordinates of Q' referred to the moving axes at $(\alpha+d\alpha, \beta+d\beta)$, and $\xi+d\xi$, $\eta+d\eta$, $\zeta+d\zeta$ the coordinates of Q' referred to the fixed axes. As before we may choose the fixed axes to coincide with the axes of x, y, z at (α, β), and then ξ, η, ζ are identical with x, y, z but

$$\left.\begin{aligned} d\xi &= A d\alpha + B d\beta \cos\chi + dx - y\delta\theta_3 + z\delta\theta_2, \\ d\eta &= \qquad\quad B d\beta \sin\chi + dy - z\delta\theta_1 + x\delta\theta_3, \\ d\zeta &= \qquad\qquad\qquad\qquad dz - x\delta\theta_2 + y\delta\theta_1 \end{aligned}\right\}.$$

As before these give the partial differential coefficients of ξ, η, ζ in terms of those of x, y, z with respect to α, β, and important formulæ may be derived by supposing that the point Q' coincides with the point Q, so that $d\xi, d\eta, d\zeta$ all vanish. In this case we find

$$\left.\begin{aligned} \frac{\partial x}{\partial \alpha} &= -A + r_1 y - q_1 z, & \frac{\partial x}{\partial \beta} &= -B\cos\chi + r_2 y - q_2 z, \\ \frac{\partial y}{\partial \alpha} &= p_1 z - r_1 x, & \frac{\partial y}{\partial \beta} &= -B\sin\chi + p_2 z - r_2 x, \\ \frac{\partial z}{\partial \alpha} &= q_1 x - p_1 y, & \frac{\partial z}{\partial \beta} &= q_2 x - p_2 y \end{aligned}\right\}.$$

Forming from these the two expressions for $\partial^2 x/\partial\alpha\partial\beta$, $\partial^2 y/\partial\alpha\partial\beta$, and $\partial^2 z/\partial\alpha\partial\beta$, and equating them, and making use of the formulæ (5), we deduce three equations, viz.

$$\left.\begin{aligned} \frac{\partial A}{\partial \beta} + r_1 B\sin\chi &= \frac{\partial}{\partial\alpha}(B\cos\chi), \\ r_1 B\cos\chi &= -\frac{\partial}{\partial\alpha}(B\sin\chi) + r_2 A, \\ -q_1 B\cos\chi + p_1 B\sin\chi &= -q_2 A \end{aligned}\right\} \quad\text{..........................} \quad (6).$$

These are equivalent to the last three of equations (6) on p. 203.

Now to find the equation for the radii of curvature, and the differential equation of the lines of curvature, we must form the conditions that the normal at $(\alpha+d\alpha, \beta+d\beta)$ intersects the normal at (α, β).

The equation of the former line referred to axes of ξ, η, ζ coinciding with the axes of x, y, z at (α, β) are

$$\frac{\xi - (A d\alpha + B d\beta\cos\chi)}{l+dl} = \frac{\eta - (B d\beta \sin\chi)}{m+dm} = \frac{\zeta}{n+dn} = \rho \text{ say},$$

where l, m, n are the direction-cosines of the normal at (α, β). Thus $\lambda = l = 0$, $\mu = m = 0$, $\nu = n = 1$, but

$$dl = \delta\theta_2, \quad dm = -\delta\theta_1, \quad dn = 0.$$

The above equation therefore is

$$\frac{\xi - (A d\alpha + B d\beta\cos\chi)}{\delta\theta_2} = \frac{\eta - (B d\beta\sin\chi)}{-\delta\theta_1} = \frac{\zeta}{1} = \rho,$$

and, if ξ, η, ζ be either of the two centres of principal curvature, ρ is the corresponding radius of curvature. The condition that the line above written may pass through a centre of principal curvature is the condition that this line intersects the line $\xi=0$, $\eta=0$, which is the normal at (α, β). Hence we find

$$\rho=-\frac{A\cdot(d\alpha/d\beta)+B\cos\chi}{q_1(d\alpha/d\beta)+q_2}=\frac{B\sin\chi}{p_1(d\alpha/d\beta)+p_2}.$$

The elimination of $d\alpha/d\beta$ gives the equation (7) of p. 203 for ρ, viz.:

$$\rho^2(p_1q_2-p_2q_1)+\rho(Bp_1\cos\chi+Bq_1\sin\chi-Ap_2)+AB\sin\chi=0 \;\ldots\ldots(7),$$

and the elimination of ρ gives the equation (8) of p. 203 for the directions of the lines of curvature, viz.:

$$B(p_2\cos\chi+q_2\sin\chi)\,d\beta^2+Ap_1d\alpha^2+(Bp_1\cos\chi+Bq_1\sin\chi+Ap_2)\,d\alpha d=0\ldots(8).$$

INDEX.

The numbers refer to pages.

END OF VOL. II.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. AND SONS, AT THE UNIVERSITY PRESS.